

CHERN CHARACTER FOR GLOBAL MATRIX FACTORIZATIONS

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ABSTRACT. We give a formula for the Chern character on the DG-category of global matrix factorizations on a smooth scheme X with superpotential $w \in \Gamma(\mathcal{O}_X)$. Our formula takes values in a Čech model for Hochschild homology. Our methods may also be adapted to get an explicit formula for the Chern character for perfect complexes of sheaves on X taking values in right derived global sections of the De-Rham algebra. Along the way we prove that the DG version of the Chern Character coincides with the classical one for perfect complexes.

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1. INTRODUCTION

Shklyarov in [14] gives a beautiful interpretation of the Chern Character and Riemann-Roch theorem in the context of DG-categories over a field k . In his treatment, he uses functoriality of Hochschild homology and the canonical functor $k_E : k \rightarrow \mathfrak{C}$, which simply sends the DG-algebra k to the object $E \in \mathfrak{C}$, to get the Chern character of E ,

$$ch(E) = HH(k_E) : k = HH(k) \rightarrow HH(\mathfrak{C}).$$

In the case when \mathfrak{C} is a proper DG-category, i.e. the diagonal bimodule, Δ , takes values in perfect k -modules ($\text{Perf } k$), we use the Kunneth isomorphism and the isomorphism $HH(\mathfrak{C}^{\text{op}}) \cong HH(\mathfrak{C})$ to obtain a pairing on homology:

$$\langle -, - \rangle_{\mathfrak{C}} : HH(\mathfrak{C}) \otimes HH(\mathfrak{C}) \cong HH(\mathfrak{C} \otimes \mathfrak{C}^{\text{op}}) \xrightarrow{HH(\Delta)} HH(\text{Perf } k) = k$$

With this pairing and definition of the Chern character, the Riemann-Roch theorem,

$$\langle ch(E), ch(F) \rangle_{\mathfrak{C}} = \text{str } \text{Hom}_{\mathfrak{C}}(E, F),$$

then becomes almost tautological, following easily from functoriality.

As with all beautiful things, the hard part is in the application. That is, for a particular DG-category, \mathfrak{C} , the difficulty is to get a meaningful handle on the Chern character and the pairing on Hochschild homology. The DG-categories of interest to us presently are certain categories of (global) matrix factorizations. We also only focus on the first half of the problem, i.e. to compute the Chern Character, taking values in some reasonable model for Hochschild homology. We, in fact, concern ourselves with a mildly more general problem:

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to compute the so called *boundary bulk map*. This is a map from the endomorphism DG-algebra of an object to Hochschild homology. We recommend [6] for details, in particular for the proof of the fact that the Chern Character is simply the boundary bulk map evaluated at the identity. Our formula for this map is rather involved, too much so to reproduce here (the impatient reader may thumb to theorem 6.5), however in the case when our matrix factorization, \mathcal{E} , admits a global connection, ∇ , i.e. global connections on graded components $\nabla_i : \mathcal{E}_i \rightarrow \Omega \otimes \mathcal{E}_i$, $i = 0, 1$, we obtain the following formula for the Chern Character:

$$ch(\mathcal{E}) = str \left(\sum_{i=0}^{\dim X} \frac{[\nabla, e]^i}{i!} \right)$$

where str denotes the super-trace, e is the curved differential on \mathcal{E} , and

$$[\nabla, e] = \nabla_{i+1}e_i - 1 \otimes e_i \nabla_i \quad i = 0, 1.$$

We save understanding the pairing for a later work.

This paper is organized as follows. Section 2 contains the background information on our particular version of matrix factorizations (taken from [5]). Section 3 contains the DG/triangulated category theory pertaining to matrix factorizations that we will need. Some of results therein have not appeared in the generality in which we state them, but by no means is anything new. In section 4, we carry out the computation of Hochschild Homology for our categories of matrix factorizations. The method for this computation is suggested in [7] and the analogous computation is carried out for Hochschild Cohomology therein. We give the details for homology. This result is also know by other methods from [10].

Sections 5 and 6 form the heart of the paper, culminating in the a formula for the boundary-bulk map which takes values in a Cech model for Hochschild Homology of matrix factorizations. This formula makes use of a choice of local connections on a Cech cover of the scheme X . In our opinion, more interesting than the formula, is the observation that the boundary-bulk map, which is a map in the derived category of complexes of vector spaces, may be promoted to the derived category of sheaves on our space X . Section 5 is concerned with understanding this promotion. Section 6 is concerned with what then happens upon applying right-derived global sections.

Throughout we assume the reader is mildly familiar with DG-categories and recommend [15] for those who are not. Our specific conventions are as follows. We fix once and for all a field k . As one particular foundational lemma (3.11) will require it, we assume that k is perfect. All categories we work with will be k -linear. In particular, “scheme” will mean k -scheme so that quasi-coherent sheaves on said scheme form a k -linear category. $C(k)$ will denote the category of chain complexes of k vector-spaces. By *modules* and $\mathfrak{C} - Mod$ we will always mean right modules, i.e. contravariant DG functors from \mathfrak{C} with values in $C(k)$. All grading will be \mathbb{Z} gradings, though often there will be a 2-periodicity among graded components. We will use cohomological grading conventions. Subscripts will not denote a change from this convention, but instead will be used to reference internal grading for objects. For example if C is a chain complex we will write C_i for the i -th graded component (the differential has $d : C_i \rightarrow C_{i+1}$), whereas we would write $\cdots \rightarrow C^{-1} \rightarrow C^0 \rightarrow C^1 \rightarrow \cdots$ for a complex of complexes with each C^j a complex in its own right. We will use $\#$ to denote the “underlying graded object” functor, which forgets any additional structure (e.g. the differential) except the \mathbb{Z} -grading. For example if

$$C = \cdots \rightarrow C_{-1} \rightarrow C_0 \rightarrow C_1 \rightarrow \cdots$$

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is a chain complex then $C^\#$ is the graded vector space

$$C^\# = \bigoplus C_i.$$

For any DG category \mathfrak{C} , we may form two regular categories $Z^0\mathfrak{C}$ and $H^0\mathfrak{C}$, with the same objects as \mathfrak{C} but with homs given by

$$\mathrm{Hom}_{Z^0\mathfrak{C}}(a, b) = Z^0\mathrm{Hom}_{\mathfrak{C}}(a, b) \quad \mathrm{Hom}_{H^0\mathfrak{C}}(a, b) = H^0\mathrm{Hom}_{\mathfrak{C}}(a, b)$$

where on the right-hand sides Z^0 and H^0 are the usual zero cycle and zero cohomology functors for chain complexes. We may also form the derived category $D\mathfrak{C}$, which is the Verdier localization of $H^0(\mathfrak{C} - \mathrm{Mod})$ with respect to quasi-isomorphisms.

2. MATRIX FACTORIZATIONS

We work with categories of matrix factorizations as in [5] and [8]. Specifically we let X be a noetherian k -scheme, \mathcal{L} a line bundle on X and $w \in \mathcal{L}(X)$ a global section. A *matrix factorization*, denoted

$$\mathcal{E} = \mathcal{E}_0 \begin{array}{c} \xrightarrow{e_0} \\ \xleftarrow{e_1} \end{array} \mathcal{E}_1$$

on X with *potential* $w \in \mathcal{L}(X)$ consists of the data of two vector-bundles \mathcal{E}_0 and \mathcal{E}_1 on X and maps

$$e_1 : \mathcal{E}_1 \rightarrow \mathcal{E}_0 \quad \text{and} \quad e_0 : \mathcal{E}_0 \rightarrow \mathcal{E}_1 \otimes \mathcal{L}$$

such that $e_0 e_1 = \mathrm{id}_{\mathcal{E}_1} \otimes w$ and $(e_1 \otimes \mathrm{id}_{\mathcal{L}}) e_0 = \mathrm{id}_{\mathcal{E}_0} \otimes w$. Twisting by \mathcal{L} and expanding 2-periodically we may view a matrix factorization as a “complex” of sheaves except the differential, e , has e^2 is multiplication by w :

$$(1) \quad \cdots \rightarrow \mathcal{E}_1 \otimes \mathcal{L}^{-1} \rightarrow \mathcal{E}_0 \otimes \mathcal{L}^{-1} \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_0 \rightarrow \mathcal{E}_1 \otimes \mathcal{L} \rightarrow \mathcal{E}_0 \otimes \mathcal{L} \rightarrow \cdots$$

Here the term $\mathcal{E}_i \otimes \mathcal{L}^k$ lives in degree $2k - i$. Given two matrix factorizations on X with potential w ,

$$\mathcal{E} = \mathcal{E}_0 \begin{array}{c} \xrightarrow{e_0} \\ \xleftarrow{e_1} \end{array} \mathcal{E}_1, \quad \mathcal{D} = \mathcal{D}_0 \begin{array}{c} \xrightarrow{d_0} \\ \xleftarrow{d_1} \end{array} \mathcal{D}_1$$

we may define a complex of morphisms $\mathrm{Hom}(\mathcal{E}, \mathcal{D})$ whose underlying graded components are

$$\mathrm{Hom}^{2k}(\mathcal{E}, \mathcal{F}) := \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_0, \mathcal{F}_0 \otimes \mathcal{L}^k) \oplus \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_1, \mathcal{F}_1 \otimes \mathcal{L}^k)$$

and

$$\mathrm{Hom}^{2k+1}(\mathcal{E}, \mathcal{F}) := \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_1, \mathcal{F}_0 \otimes \mathcal{L}^k) \oplus \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_0, \mathcal{F}_1 \otimes \mathcal{L}^{k+1}).$$

The differential on $\mathrm{Hom}(\mathcal{E}, \mathcal{D})$ is given by $\partial(f) = df - (-1)^{|f|} f e$. One easily verifies that $\partial^2 = 0$ so $\mathrm{Hom}(\mathcal{E}, \mathcal{D})$ is indeed an honest complex, even though \mathcal{E} and \mathcal{D} are not.

The DG-category of matrix factorizations defined above is not “correct” in the global setting. It contains objects which “should be” 0 in the DG-derived category but are not, i.e. there are locally contractible matrix factorizations which are not globally contractible. There are several ways of dealing with this. In [4] Orlov defines the derived category of matrix factorizations to be the Verdier quotient of the derived category of matrix factorizations in standard DG sense by the thick subcategory of locally contractible objects. Alternatively one can form a new DG-category $\mathfrak{MF}_{loc}(X, \mathcal{L}, w)$, in which we localize with respect to the spacial variable. The objects of $\mathfrak{MF}_{loc}(X, \mathcal{L}, w)$ are the same as in $\mathfrak{MF}(X, \mathcal{L}, w)$ and morphisms are given by a suitably functorial models (so that

composition is well-defined) for the complexes $\mathbb{R}\mathrm{Hom}(\mathcal{E}_i, \mathcal{F}_j \otimes \mathcal{L}^n)$ for $i, j = 0, 1$ and all n , then defining the morphism complex in the “corrected” category to be

$$\mathrm{Hom}_{\mathfrak{M}\mathfrak{F}_{loc}(X, \mathcal{L}, w)}(\mathcal{E}, \mathcal{F}) := \mathrm{Tot}(\mathbb{R}\mathrm{Hom}(\mathcal{E}, \mathcal{F}))$$

This can be done using a Cech model as in [7] and [13] or by choosing functorial injective resolutions which we explain below.

In what follows we will want to consider a slightly larger class of objects obtained by dropping the restriction that \mathcal{E}_0 and \mathcal{E}_1 be vector bundles and allowing the graded components of \mathcal{E} to arbitrary quasi-coherent sheaves on X . We will refer to such an object as a *curved quasi-coherent \mathcal{O}_X module*. The DG-category of curved modules with Hom complexes defined above will be denoted by $\mathfrak{Q}\mathrm{coh}(X, \mathcal{L}, w)$, the full subcategory of matrix factorizations will be denoted by $\mathfrak{M}\mathfrak{F}(X, \mathcal{L}, w)$ and the full subcategory formed by considering curved sheaves with coherent graded components will be $\mathfrak{C}\mathrm{oh}(X, \mathcal{L}, w)$. We will often drop X and or \mathcal{L} from the notation, when they are clear from context. We will keep w , to distinguish $\mathfrak{C}\mathrm{oh}(w)$ (resp. $\mathfrak{Q}\mathrm{coh}(w)$) from the categories $\mathbf{Coh}(X)$ (resp. $\mathbf{Qcoh}(X)$) of ordinary (quasi-)coherent sheaves on X .

In [9] Positselski defines the notion of a *curved differential graded ring* (CDG-ring) as a Graded ring $B = \bigoplus B_i$ along with a degree 1 endomorphism d and an element $w \in B_2$ such that $\delta^2 = [w, -]$. A B -module is a graded (left) $B^\#$ -module M endowed with its own differential d_M satisfying the compatibility identity

$$d_M(am) = d(a)m - (-1)^{|a|}ad_M(m).$$

Morphisms between curved modules are $B^\#$ module morphisms and are endowed with a differential in the standard way. As with matrix factorizations this differential produces a complex.

As in [8] we may use a sheafified version of curved modules to describe the category $\mathfrak{Q}\mathrm{coh}(w)$. We define a sheaf of CDG-rings $S(\mathcal{L}) = \bigoplus_{i \in \mathbb{Z}} \mathcal{L}^i$ as the “free algebra” on \mathcal{L} , graded such that \mathcal{L} lives in degree 2 and we endow $S(\mathcal{L})$ with the trivial differential. Then a curved \mathcal{O}_X module with potential $w \in \mathcal{L}(X)$ is a Quasi-coherent CDG $S(\mathcal{L})$ -module, i.e. a curved quasi-coherent module as defined above gives rise to a \mathbb{Z} -graded $S(\mathcal{L})$ -module

$$\dots \xrightarrow{e_1} \mathcal{E}_0 \otimes \mathcal{L}^{-1} \xrightarrow{e_0} \mathcal{E}_1 \rightarrow \mathcal{E}_0 \xrightarrow{e_1} \mathcal{E}_1 \otimes \mathcal{L}^1 \xrightarrow{e_0} \mathcal{E}_0 \otimes \mathcal{L}^2 \xrightarrow{e_1} \dots$$

with differential e such that e^2 is multiplication by w . One can check that the morphisms in $\mathfrak{Q}\mathrm{coh}(w)$ are precisely the morphisms of CDG $S(\mathcal{L})$ -modules, i.e. graded-morphisms on the underlying \mathcal{O}_X modules which commute with the $S(\mathcal{L})$ action.

Conversely given a curved $S(\mathcal{L})$ -module $(\mathcal{M}, d_{\mathcal{M}})$, the natural isomorphisms

$$\mathcal{L}^{-1} \otimes \mathcal{L} \cong \mathcal{O}_X \cong \mathcal{L} \otimes \mathcal{L}^{-1}$$

and the associativity of multiplication imply that any $S(\mathcal{L})$ -module, \mathcal{M} , must have isomorphisms

$$\mathcal{M}_i \otimes \mathcal{L} \cong \mathcal{M}_{i+2}$$

for all i . This gives an equivalence of categories between the $\mathfrak{Q}\mathrm{coh}(w)$ and the category of CDG curved $S(\mathcal{L})$ modules with curvature w . We will use both interpretations of $\mathfrak{Q}\mathrm{coh}(X, \mathcal{L}, w)$ interchangeably. We will continue to use the notation $\mathfrak{Q}\mathrm{coh}(X, \mathcal{L}, w)$ for both.

For curved modules $\mathcal{M} \in \mathfrak{C}\mathrm{oh}(X, \mathcal{L}, w)$ and $\mathcal{N} \in \mathfrak{Q}\mathrm{coh}(X, \mathcal{L}, w')$ we may form the curved module $\mathcal{H}om_{S(\mathcal{L})}(\mathcal{M}, \mathcal{N}) \in \mathfrak{Q}\mathrm{coh}(X, \mathcal{L}, w' - w)$ defined by

$$\mathcal{H}om_{S(\mathcal{L})}(\mathcal{M}, \mathcal{N})^\# = \mathcal{H}om_{S(\mathcal{L})^\#}(\mathcal{M}^\#, \mathcal{N}^\#)$$

and whose differential is given by $\partial(f) = d_{M'}f - (-1)^{|f|}fd_{\mathcal{N}}$. In particular for $\mathcal{M} \in \mathfrak{Coh}(X, \mathcal{L}, w)$ we have the dual module

$$\mathcal{M}^\vee = \mathcal{H}om_{S(\mathcal{L})}(\mathcal{M}, S(\mathcal{L})) \in \mathfrak{Coh}(X, \mathcal{L}, -w).$$

For $\mathcal{M} \in \mathfrak{Qcoh}(X, \mathcal{L}, w)$ and $\mathcal{N} \in \mathfrak{Qcoh}(X, \mathcal{L}, w')$ we may form the tensor product $\mathcal{M} \otimes_{S(\mathcal{L})} \mathcal{N} \in \mathfrak{Qcoh}(X, \mathcal{L}, w + w')$ defined by

$$(\mathcal{M} \otimes_{S(\mathcal{L})} \mathcal{N})^\# = \mathcal{M}^\# \otimes_{S(\mathcal{L})^\#} \mathcal{N}^\#$$

and whose differential is given by $d_{\mathcal{M} \otimes \mathcal{N}} = d_{\mathcal{M}} \otimes 1 + 1 \otimes d_{\mathcal{N}}$. More explicitly for

$$\mathcal{M} = \mathcal{M}_0 \begin{array}{c} \xrightarrow{m_0} \\ \xleftarrow{m_1} \end{array} \mathcal{M}_1, \quad \mathcal{N} = \mathcal{N}_0 \begin{array}{c} \xrightarrow{n_0} \\ \xleftarrow{n_1} \end{array} \mathcal{N}_1$$

we have

$$\begin{array}{ccc} \mathcal{H}om_{S(\mathcal{L})}(\mathcal{M}, \mathcal{N}) = & & \\ & \begin{array}{c} \left(\begin{array}{cc} -n_{0*} & m_{0*}^* \\ m_{1*}^* & -n_{1*} \end{array} \right) \\ \xrightarrow{\quad} \\ \mathcal{H}om(\mathcal{M}_0, \mathcal{N}_0) \oplus \mathcal{H}om(\mathcal{M}_1, \mathcal{N}_1) & & \mathcal{H}om(\mathcal{M}_0, \mathcal{N}_1) \oplus \mathcal{H}om(\mathcal{M}_1, \mathcal{N}_0) \\ \xleftarrow{\quad} \\ \left(\begin{array}{cc} n_{1*} & m_{1*}^* \\ m_{0*}^* & n_{0*} \end{array} \right) \end{array} & & \end{array}$$

and

$$\begin{array}{ccc} \mathcal{M} \otimes_{S(\mathcal{L})} \mathcal{N} = & & \\ & \begin{array}{c} \left(\begin{array}{cc} n_{0*} \otimes 1 & 1 \otimes m_{0*} \\ 1 \otimes m_{1*} & n_{1*} \otimes 1 \end{array} \right) \\ \xrightarrow{\quad} \\ \mathcal{M}_0 \otimes \mathcal{N}_0 \oplus \mathcal{M}_1 \otimes \mathcal{N}_1 & & \mathcal{M}_0 \otimes \mathcal{N}_1 \oplus \mathcal{M}_1 \otimes \mathcal{N}_0 \\ \xleftarrow{\quad} \\ \left(\begin{array}{cc} n_{1*} \otimes 1 & 1 \otimes m_{1*} \\ 1 \otimes m_{0*} & n_{0*} \otimes 1 \end{array} \right) \end{array} & & \end{array}$$

The following facts about the $\mathcal{H}om_{S(\mathcal{L})}$ and $\otimes_{S(\mathcal{L})}$ functors are easily verified by sheafifying the natural isomorphisms that arise when X is affine.

Proposition 2.1. *Let $\mathcal{M} \in \mathfrak{Coh}(X, \mathcal{L}, w)$, $\mathcal{N} \in \mathfrak{Coh}(X, \mathcal{L}, v)$, $\mathcal{P} \in \mathfrak{Qcoh}(X, \mathcal{L}, u)$ and $\mathcal{E} \in \mathfrak{Mf}(X, \mathcal{L}, t)$ and $\mathcal{D} \in \mathfrak{Mf}(X, \mathcal{L}, s)$ then*

(1) $\mathcal{H}om_{S(\mathcal{L})}(\mathcal{M} \otimes_{S(\mathcal{L})} \mathcal{N}, \mathcal{P}) \cong \mathcal{H}om_{S(\mathcal{L})}(\mathcal{M}, \mathcal{H}om_{S(\mathcal{L})}(\mathcal{N}, \mathcal{P}))$ naturally as objects of $Z^0\mathfrak{Qcoh}(X, \mathcal{L}, u - v - w)$.

(2) $\mathcal{H}om_{S(\mathcal{L})}(\mathcal{M} \otimes_{S(\mathcal{L})} \mathcal{E}, \mathcal{N}) \cong \mathcal{H}om_{S(\mathcal{L})}(\mathcal{M}, \mathcal{E}^\vee \otimes_{S(\mathcal{L})} \mathcal{N}) \cong \mathcal{H}om_{\mathfrak{S}(L)}(\mathcal{M}, \mathcal{N}) \otimes_{S(\mathcal{L})} \mathcal{E}^\vee$ naturally as objects of $Z^0\mathfrak{Qcoh}(X, \mathcal{L}, v - w - t)$.

(3) $\mathcal{E}^\vee \otimes_{S(\mathcal{L})} \mathcal{D} \cong \mathcal{H}om_{S(\mathcal{L})}(\mathcal{E}, \mathcal{D})$ as objects of $Z^0\mathfrak{Qcoh}(X, \mathcal{L}, s - t)$.

(4) $(\mathcal{E}^\vee)^\vee \cong \mathcal{E}$ naturally in $Z^0\mathfrak{Mf}(X, \mathcal{L}, t)$ and the functor

$$\vee : \mathfrak{Mf}(X, \mathcal{L}, t)^{op} \rightarrow \mathfrak{Mf}(X, \mathcal{L}, -t)$$

is an equivalence of DG-categories.

The category $Z^0\mathfrak{Qcoh}(w)$ is easily seen to be an abelian category with arbitrary direct sums. From [8] the homotopy category $H^0\mathfrak{Qcoh}(w)$ admits a triangulated structure with the obvious shift functor and in which distinguished triangles are isomorphic to triangles of the form

$$\mathcal{E} \xrightarrow{f} \mathcal{D} \rightarrow Cone(f) \rightarrow \mathcal{E}[1]$$

where $Cone(f)$ is defined analogously to cones in the category of complexes of sheaves. Also, as we will use it frequently, if

$$\dots \rightarrow \mathcal{M}^{-1} \rightarrow \mathcal{M}^0 \rightarrow \mathcal{M}^1 \rightarrow \dots$$

is a complex of curved modules (where the curved modules are viewed in the abelian category $Z^0\mathfrak{Qcoh}(w)$) we may form the direct sum total curved module $Tot(\mathcal{M}^\bullet)$ whose graded components are $Tot(\mathcal{M}^\bullet)^n = \bigoplus_{p+q=n} \mathcal{M}_q^p$ and whose curved differential is given by the formula analogous to forming the total complex for complexes of sheaves.

The functor $\# : Z^0\mathfrak{Qcoh}(w) \rightarrow S(\mathcal{L})^\# - Mod_0$, where $S(\mathcal{L})^\# - Mod_0$ denotes the category of $S(\mathcal{L})^\#$ modules with degree 0 morphisms, admits left and right adjoints $+$ and $-$ defined by

$$\mathcal{M}^+ = \mathcal{M}_0 \oplus \mathcal{M}_1 \begin{array}{c} \xrightarrow{\begin{pmatrix} 0 & 1 \\ w & 0 \end{pmatrix}} \\ \xleftarrow{\begin{pmatrix} 0 & w \\ 1 & 0 \end{pmatrix}} \end{array} \mathcal{M}_1 \otimes \mathcal{L}^{-1} \oplus \mathcal{M}_0$$

and

$$\mathcal{M}^- = \mathcal{M}_0 \oplus \mathcal{M}_1 \otimes \mathcal{L} \begin{array}{c} \xrightarrow{\begin{pmatrix} 0 & w \\ 1 & 0 \end{pmatrix}} \\ \xleftarrow{\begin{pmatrix} 0 & 1 \\ w & 0 \end{pmatrix}} \end{array} \mathcal{M}_1 \oplus \mathcal{M}_0$$

Evidently the functors $+$ and $-$ are exact.

We may use these adjoints to construct right and left resolutions in the abelian $Z^0\mathfrak{Qcoh}(w)$, by first resolving as graded sheaves of $S(\mathcal{L})^\#$ -modules and then applying either $+$ or $-$ appropriately. Specifically when

$$(\mathcal{F}^\bullet)^\# \rightarrow \mathcal{E}^\#$$

is a resolution of \mathcal{E} as a graded $S(\mathcal{L})$ module then

$$((\mathcal{F}^\bullet)^\#)^+ \rightarrow \mathcal{E}$$

resolves \mathcal{E} as a w -curved $S(\mathcal{L})$ modules and similarly when

$$\mathcal{E}^\# \rightarrow (\mathcal{I}^\bullet)^\#$$

resolves \mathcal{E} then

$$\mathcal{E} \rightarrow ((\mathcal{I}^\bullet)^\#)^-$$

is a resolution as w -curved modules. We will be particularly interested in the cases when $(\mathcal{F}^\bullet)^\#$ consists of flat sheaves, vector bundles, or locally free sheaves and when $(\mathcal{I}^\bullet)^\#$ consists of injective sheaves.

3. THE CODERIVED CATEGORY

We have yet to explain how we are to deal with locally contractible matrix factorizations or to justify our allegation that it is useful to pass to the larger category $\mathfrak{Qcoh}(X, \mathcal{L}, w)$.

As we mentioned above $H^0\mathfrak{Qcoh}(w)$ is triangulated and this category, along with its triangulated structure, is reminiscent of homotopy category of complexes of quasi-coherent sheaves on X . As such, one is interested in localizing with respect to the “acyclic” objects, which would in particular kill the locally contractible matrix factorizations. The problem is that the usual notion of “acyclic” has no obvious analog in $\mathfrak{Qcoh}(X, \mathcal{L}, w)$ unless $w = 0$. It turns out that the appropriate thing to do is to consider the exotic derived categories defined in [9], in particular the so-called *coderived* category.

Definition 3.1. We say that a curved module $\mathcal{M} \in H^0\mathfrak{Qcoh}(X, \mathcal{L}, w)$ is *coacyclic*, if \mathcal{M} is contained in the smallest triangulated category which contains the total curved modules

$$\mathrm{Tot}(\mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C})$$

for all short exact sequences in $Z^0\mathfrak{Qcoh}(w)$ and which is in addition closed under arbitrary direct sums. We will denote the triangulated category of coacyclic objects by $Coac(X, \mathcal{L}, w)$.

Definition 3.2. The coderived category of $\mathfrak{Qcoh}(X, \mathcal{L}, w)$, denoted $D^{co}(\mathfrak{Qcoh}(X, \mathcal{L}, w))$ is the Verdier quotient

$$D^{co}(\mathfrak{Qcoh}(X, \mathcal{L}, w)) = H^0\mathfrak{Qcoh}(X, \mathcal{L}, w)/Coac(X, \mathcal{L}, w)$$

We will call the morphisms of $Z^0\mathfrak{Qcoh}(X, \mathcal{L}, w)$ and $H^0\mathfrak{Qcoh}(X, \mathcal{L}, w)$ which become isomorphisms in the coderived categories weak equivalences.

Remark 3.3. One can easily check using induction that the coacyclic modules contain the total curved modules of arbitrarily long, but finite, exact sequences. In particular a curved module \mathcal{E} is weakly equivalent to any of its finite right or left resolutions. The following lemma and corollary show that \mathcal{E} is in fact weakly equivalent to any of its *infinite* right resolutions.

Lemma 3.4. *The category $Coac(X, \mathcal{L}, w)$ is closed under taking directed homotopy colimits.*

Proof. This is true of any triangulated category which is closed under taking direct sums, since the homotopy colimit is computed as the cone over a particular map between the direct sums of the objects. \square

Corollary 3.5. *If*

$$\mathcal{E} \rightarrow \mathcal{I}_1 \rightarrow \mathcal{I}_2 \dots$$

is an exact sequence in $Z^0\mathfrak{Qcoh}(X, \mathcal{L}, w)$ then the canonical map $\mathcal{E} \rightarrow \mathrm{Tot}(\mathcal{I}_\bullet)$ is a weak equivalence.

Proof. We can identify

$$Cone(\mathcal{E} \rightarrow \mathrm{Tot}(\mathcal{I}^\bullet)) = \mathrm{Tot}(\mathcal{E} \rightarrow \mathcal{I}^\bullet) = \mathrm{hocolim}_n \mathrm{Tot}(\tau^n(\mathcal{E} \rightarrow \mathcal{I}^\bullet))$$

where τ^n denotes the canonical truncation. Each $\mathrm{Tot}(\tau^n(\mathcal{E} \rightarrow \mathcal{I}^\bullet))$ is the total complex of a finite exact sequence of curved modules therefore is in $Coac(X, w, \mathcal{L})$ (cf. Remark 3.3) and then by the lemma the homotopy colimit is coacyclic as well. \square

The above corollary and the discussion involving the existence of injective resolutions in $Z^0\mathfrak{Qcoh}(X, \mathcal{L}, w)$ which concluded the previous section essentially give the following corollary.

Corollary 3.6. *For every $\mathcal{M} \in H^0\mathfrak{Qcoh}(X, \mathcal{L}, w)$ there is a triangle*

$$\mathcal{A} \rightarrow \mathcal{M} \rightarrow \mathcal{I} \rightarrow \mathcal{A}[1]$$

where \mathcal{A} is coacyclic and \mathcal{I} is graded-injective. In particular the coderived category is equivalent to the homotopy category of graded-injective curved modules.

Proof. The first claim follows from the existence of injective replacements. That the coderived category is equivalent to the homotopy category of injective modules will follow from the general theory of Verdier localization (see for example [9]) provided we can prove that $\mathrm{Hom}(\mathcal{C}, \mathcal{I})$ is acyclic whenever \mathcal{C} is coacyclic and \mathcal{I} is graded injective. For this it suffices to consider the case when \mathcal{C} is the total curved module of a short exact sequence of curved modules, for which the statement is obvious. \square

Corollary 3.6 tells us how to compute the homs in the coderived category: we pick some graded-injective replacements \mathcal{I} of \mathcal{M} and \mathcal{J} of \mathcal{N} then

$$\mathrm{Hom}_{D^{co}\mathfrak{Qcoh}(X, \mathcal{L}, w)}(\mathcal{M}, \mathcal{N}) = H^0\mathrm{Hom}_{\mathfrak{Qcoh}(X, \mathcal{L}, w)}(\mathcal{I}, \mathcal{J}).$$

The category $\mathfrak{Qcoh}(X)$ of quasi-coherent sheaves on X is a Grothendieck category and therefore admits functorial injective resolutions. Such a functor can be used to a functorial injective replacement (by simply taking the direct sum total complex) $K : \mathfrak{Qcoh}(X, \mathcal{L}, w) \rightarrow \mathrm{Inj}(X, \mathcal{L}, w)$, where $\mathrm{Inj}(X, \mathcal{L}, w)$ denotes the full DG subcategory of $\mathfrak{Qcoh}(X, \mathcal{L}, w)$ formed by the objects whose graded components are injective as \mathcal{O}_X modules. We will denote by $\mathfrak{MF}_{loc}(X, \mathcal{L}, w)$ the full subcategory of $\mathrm{Inj}(X, \mathcal{L}, w)$ formed by (the images of) matrix factorizations. We will often simply write $\mathbb{R}\mathrm{Hom}(\mathcal{E}, \mathcal{F})$ for

$$\mathrm{Hom}_{\mathfrak{MF}_{loc}(X, \mathcal{L}, w)}(\mathcal{E}, \mathcal{F}) = \mathrm{Hom}_{\mathfrak{Qcoh}(X, \mathcal{L}, w)}(K(\mathcal{E}), K(\mathcal{F})).$$

We will make use of [8] Corollary 2.3(1):

Proposition 3.7 (Positselski). *The image of the category $\mathfrak{Coh}(X, \mathcal{L}, w)$ forms a set of compact generators in $D^{co}\mathfrak{Qcoh}(X, \mathcal{L}, w)$.*

This proposition has as an immediate corollary :

Corollary 3.8. *Suppose X is smooth. The image of $\mathfrak{MF}(X, \mathcal{L}, w)$ forms a set of compact generators in $D^{co}\mathfrak{Qcoh}(X, \mathcal{L}, w)$.*

Proof. We need only to show that matrix factorizations generate since proposition 3.7 already implies that they are compact. Since X is smooth, for a coherent curved module \mathcal{M} we may form a finite resolution of $\mathcal{M}^\#$ by a complex of sheaves whose graded components are vector bundles. Applying the $+$ functor yields a resolution of \mathcal{M} by matrix factorizations. Then \mathcal{M} is weakly equivalent to the matrix factorization obtained by taking the total curved module of this resolution (c.f. once more remark 3.3). \square

This corollary then gives us an important characterization of the category of modules for $\mathfrak{MF}_{loc}(X, \mathcal{L}, w)$ and in particular justifies our claim that is useful to expand our view to the whole category of quasi-coherent curved modules:

Theorem 3.9. *The functor $\mathcal{M} \mapsto \mathrm{Hom}(-, \mathcal{M})|_{\mathfrak{MF}_{loc}}$ induces a triangulated equivalence between $D^{co}\mathfrak{Qcoh}(X, \mathcal{L}, w)$ and $D(\mathfrak{MF}_{loc}(X, \mathcal{L}, w))$*

Proof. In light of corollary 3.8, this is an application of theorem 5.1 from [2]. \square

The following theorem is announced in [7], in the case when $\mathcal{L} = \mathcal{O}_X$. We include a few details to the proof.

Theorem 3.10. *Assume that w is not a zero divisor, i.e. the map $\mathcal{O}_X \xrightarrow{w} \mathcal{L}$ is injective and that X is smooth, then the category $\mathfrak{MF}_{loc}(X, \mathcal{L}, w)$ has a compact generator.*

Proof. We use the global version of Orlov's theorem given as the Main Theorem (2.7) from [8] to get an equivalence

$$\mathfrak{Coh}(X, \mathcal{L}, w)/Coac(X, \mathcal{L}, w) \cap \mathfrak{Coh}(X, \mathcal{L}, w) \cong D_{Sing}^b(X_0/X)$$

where X_0 is closed subscheme defined by $w = 0$ and $D_{Sing}^b(X_0/X)$ is the relative singularity category defined in *Loc. Cit.* As a piece of notation we will set

$$D^{abs}\mathfrak{Coh}(X, \mathcal{L}, w) := \mathfrak{Coh}(X, \mathcal{L}, w)/Coac(X, \mathcal{L}, w) \cap \mathfrak{Coh}(X, \mathcal{L}, w).$$

By Rouquier theorem 7.39 [12] the bounded derived category of coherent sheaves on X_0 has a classical generator, \mathcal{G} . This classical generator then descends to a classical generator for the quotient $\mathcal{D}_{Sing}^b(X_0/X)$ and therefore gives a classical generator (which we will also call \mathcal{G}) for the category $D^{abs}(\mathfrak{Coh}(X, \mathcal{L}, w))$. By corollary 3.8 since \mathcal{G} is coherent, there is a weak equivalence between \mathcal{G} and some matrix factorization $\mathcal{E}_{\mathcal{G}}$. By corollary 3.8, $D^{abs}\mathfrak{Coh}(X, \mathcal{L}, w)$ generates $D^{co}\mathfrak{Qcoh}(X, \mathcal{L}, w)$, and therefore $\mathcal{E}_{\mathcal{G}}$ also generates $D^{co}\mathfrak{Qcoh}(X, \mathcal{L}, w)$. Applying the injective replacement functor $K : \mathfrak{Qcoh}(X, \mathcal{L}, w) \rightarrow Inj(X, \mathcal{L}, w)$, and using Corollary 3.6 we get $K(\mathcal{E}_{\mathcal{G}})$ is a compact generator for $Inj(X, \mathcal{L}, w)$. By definition $K(\mathcal{E}_{\mathcal{G}})$ lies in $\mathfrak{MF}_{loc}(X, \mathcal{L}, w)$ therefore is a compact generator for $\mathfrak{MF}_{loc}(X, \mathcal{L}, w)$ as well. \square

Lemma 3.11. *Assume X is a smooth k -scheme, with k a perfect field. Let $w \in \mathcal{O}_X$ and let \tilde{w} denote the doubled potential $\tilde{w} = p_1^*(w) - p_2^*(w)$ on $X \times X$. The exterior product induces a quasi-equivalence*

$$\mathfrak{MF}_{loc}(X, \mathcal{O}_X, w) \otimes \mathfrak{MF}_{loc}(X, \mathcal{O}_X, w)^{op} - Mod \cong \mathfrak{MF}_{loc}(X \times X, \mathcal{O}_{X \times X}, \tilde{w}) - Mod$$

*and under this isomorphism the diagonal bimodule corresponds to the diagonal curved module $\Delta_*S(\mathcal{O}_X)$.*

Proof. This follows from the same arguments as [7] theorem 3.4. \square

We recall from [6] for a DG-category, \mathfrak{C} , we have the trace functor

$$Tr : D(\mathfrak{C} \otimes C^{op} - Mod) \rightarrow D(C(k))$$

given by $Tr(M) := M \otimes_{C \otimes C^{op}} \Delta$, where Δ is the diagonal bimodule $\Delta(a, b) = \text{Hom}_{\mathfrak{C}}(b, a)$.

Then by [15] Hochschild homology is computed as $Tr(\Delta)$.

Lemma 3.12. *The isomorphism $D(\mathfrak{MF}_{loc}(X \times X, \tilde{w})) \cong D(\mathfrak{MF}_{loc}(X, w) \otimes \mathfrak{MF}_{loc}(X, w)^{op})$ from Lemma 3.11 followed by the trace functor is quasi-isomorphic to the functor*

$$\mathbb{R}\Gamma(\mathbb{L}\Delta^* -).$$

Proof. Both Tr and $\mathbb{R}\Gamma(\mathbb{L}\Delta^* -)$ are triangulated functors from

$$D(\mathfrak{MF}(X \times X, \tilde{w}))$$

to $C(k)$ that commute with arbitrary direct sums, so it will suffice to check that they give the same result at the compact generator of $D(\mathfrak{MF}(X \times X, \tilde{w}))$. For this we compute

$$\mathbb{R}\Gamma(\mathbb{L}\Delta^* \mathcal{E} \boxtimes \mathcal{F}^\vee) = \mathbb{R}\Gamma(\mathcal{E} \otimes \mathcal{F}^\vee) = \mathbb{R}\text{Hom}(\mathcal{F}, \mathcal{E}).$$

\square

4. HOCHSCHILD HOMOLOGY

In this section we compute the Hochschild homology of the category of matrix factorizations in the case when $\mathcal{L} = \mathcal{O}_X$. In fact, from now on all of our results will apply only to the case $\mathcal{L} = \mathcal{O}_X$, we save the more general case for later work. We will also assume now on that X is smooth. We follow very closely the computation of Hochschild cohomology which appears in [7]. An alternative computation appears in [10] and at this point this result is well-known to the experts. We include our computation for completeness and since we will later have use to examine more closely the particular isomorphisms needed to compare the Hochschild homology to a certain complex involving forms on X .

Following [7], we define the *complete bar complex* $\widehat{\mathcal{B}ar}$. This complex has graded components $\widehat{\mathcal{B}ar}_{-q} = (p_{1,q+2})_* \mathcal{O}_{\mathfrak{X}^{q+2}}$ for $q \geq 0$, where \mathfrak{X}^k is the completion of

$$X^k = X \times \cdots \times X$$

along the diagonal and $p_{1,q+2} : X \times \cdots \times X \rightarrow X \times X$ projects to the first and last factor in the obvious way. To reduce clutter with our notation, we will hence forth simply write $\mathcal{O}_{\mathfrak{X}^k}$, rather than the push forward onto the first and last factor. The reader hopefully will keep in mind that $\mathcal{O}_{\mathfrak{X}^k}$ is actually viewed as a sheaf on $X \times X$.

The differential,

$$b : \widehat{\mathcal{B}ar}_{-q} \rightarrow \widehat{\mathcal{B}ar}_{-q+1}$$

is given locally by the standard formula for the bar differential:

$$b(a_0 \boxtimes \cdots \boxtimes a_{q+1}) = \sum_{i=0}^q (-1)^i a_0 \boxtimes \cdots \boxtimes a_i a_{i+1} \boxtimes \cdots \boxtimes a_{q+1}.$$

Here (and elsewhere) we use \boxtimes to emphasize that this is an external tensor (i.e) only scalars commute with it as opposed to a tensor over \mathcal{O}_X . We introduce a new “differential” of degree -1, B_w , defined locally by the equation

$$B_w(a_0 \boxtimes \cdots \boxtimes a_{q+1}) = \sum_{i=0}^q (-1)^i a_0 \boxtimes \cdots \boxtimes a_i \boxtimes w \boxtimes a_{i+1} \boxtimes \cdots \boxtimes a_{q+1}.$$

We now define the *curved complete bar complex*, $\widehat{\mathcal{B}ar}_{\tilde{w}}$, as follows. This will be an object of $\mathfrak{Qcoh}(X \times X, \mathcal{O}_{X \times X}, \tilde{w})$, where once again $\tilde{w} = p_1^*(w) - p_2^*(w)$ and $p_i : X \times X \rightarrow X$ are the standard projections. Again this follows [7].

We put

$$(\widehat{\mathcal{B}ar}_{\tilde{w}})_q = \bigoplus_{p \equiv q \pmod{2}} \widehat{\mathcal{B}ar}_{-p}.$$

The map B_w may be viewed as a map of degree 1 in $\widehat{\mathcal{B}ar}_{\tilde{w}}$ by mapping the factor $(\widehat{\mathcal{B}ar}_{\tilde{w}})_p$ in $(\widehat{\mathcal{B}ar}_{\tilde{w}})_q$ to $(\widehat{\mathcal{B}ar}_{\tilde{w}})_{-(p+1)}$ in $(\widehat{\mathcal{B}ar}_{\tilde{w}})_{q+1}$. We imbue $\widehat{\mathcal{B}ar}_{\tilde{w}}$ with the curved differential $\partial = b + B_w$, then one checks that $B_w^2 = 0$ and then that

$$\partial^2 = bB_w + B_w b = \tilde{w}$$

so $\widehat{\mathcal{B}ar}_{\tilde{w}}$ is indeed a \tilde{w} -curved module.

It is helpful to view $\widehat{\mathcal{B}ar}_{\tilde{w}}$ as the total complex (perhaps modulo some signs) of the following “bi-complex”:

$$\begin{array}{ccccccc}
& & \vdots & & \vdots & & \vdots \\
& & \nwarrow B_w & & \nwarrow B_w & & \nwarrow B_w \\
\dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
& & \nwarrow B_w & & \nwarrow B_w & & \nwarrow B_w \\
\dots & \xrightarrow{b} & \mathcal{O}_{\mathfrak{X}^4} & \xrightarrow{b} & \mathcal{O}_{\mathfrak{X}^3} & \xrightarrow{b} & \mathcal{O}_{\mathfrak{X}^2} \\
& & \nwarrow B_w & & \nwarrow B_w & & \nwarrow B_w \\
\dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
& & \nwarrow B_w & & \nwarrow B_w & & \nwarrow B_w \\
\dots & \xrightarrow{b} & \mathcal{O}_{\mathfrak{X}^4} & \xrightarrow{b} & \mathcal{O}_{\mathfrak{X}^3} & \xrightarrow{b} & \mathcal{O}_{\mathfrak{X}^2} \\
& & \nwarrow B_w & & \nwarrow B_w & & \nwarrow B_w \\
\dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
& & \vdots & & \vdots & & \vdots
\end{array}$$

There is a map $\widehat{\mathcal{B}ar}_{\tilde{w}}(X) \xrightarrow{\epsilon} \Delta_* S(\mathcal{O}_X)$ given by projecting the even components onto $\mathcal{O}_{\mathfrak{X}^2}$ then using the multiplication map

$$\mathcal{O}_{\mathfrak{X}^2} \rightarrow \mathcal{O}_{\Delta}$$

and sending the odd components to 0. It is easy to check that this defines a closed degree 0 morphism of curved complexes.

Lemma 4.1. $\widehat{\mathcal{B}ar}_{\tilde{w}} \otimes_{S(\mathcal{O}_{X \times X})} \mathcal{M}$ is isomorphic to $\Delta \otimes_{S(\mathcal{O}_{X \times X})}^{\mathbb{L}} \mathcal{M}$ in $D(X)$ for any $- \tilde{w}$ curved module \mathcal{M} .

Proof. We let \mathcal{W} be the cone of the morphism $\epsilon : \widehat{\mathcal{B}ar}_{\tilde{w}}(X) \rightarrow \Delta_* S(\mathcal{O}_X)$. Then

$$\mathcal{W}_n = \bigoplus_{k \equiv n \pmod{2}} \mathcal{O}_{\mathfrak{X}^k}$$

where we consider $\mathfrak{X}^1 = \Delta$.

Consider first the case when \mathcal{M} is graded-flat. The n -th graded component of the complex $\mathcal{W} \otimes_{S(\mathcal{O}_{X \times X})} \mathcal{M}$ is

$$\begin{aligned}
(\mathcal{W} \otimes_{S(\mathcal{O}_{X \times X})} \mathcal{M})_n &= \bigoplus_{k \equiv n \pmod{2}} \mathcal{O}_{\mathfrak{X}^k} \otimes \mathcal{M}_0 \oplus \bigoplus_{k \equiv n+1 \pmod{2}} \mathcal{O}_{\mathfrak{X}^k} \otimes \mathcal{M}_1 \\
&= \bigoplus_k \mathcal{O}_{\mathfrak{X}^k} \otimes \mathcal{M}_{n-k}
\end{aligned}$$

Taking the differential into account, may view $\mathcal{W} \otimes_{S(\mathcal{O}_{X \times X})} \mathcal{M}$ as the total complex of the “bi-complex”

$$\begin{array}{ccccccc}
& & \vdots & \nearrow & \vdots & \nearrow & \vdots & \nearrow & \vdots \\
& & \vdots & \nearrow & \vdots & \nearrow & \vdots & \nearrow & \vdots \\
\cdots & \longrightarrow & \mathcal{O}_{\mathfrak{X}^4} \otimes \mathcal{M}_1 & \longrightarrow & \mathcal{O}_{\mathfrak{X}^3} \otimes \mathcal{M}_1 & \longrightarrow & \mathcal{O}_{\mathfrak{X}^2} \otimes \mathcal{M}_1 & \xrightarrow{\epsilon} & \mathcal{O}_{\Delta} \otimes \mathcal{M}_1 \\
& \nearrow & \uparrow & \nearrow & \uparrow & \nearrow & \uparrow & \nearrow & \uparrow \\
\cdots & \longrightarrow & \mathcal{O}_{\mathfrak{X}^4} \otimes \mathcal{M}_0 & \longrightarrow & \mathcal{O}_{\mathfrak{X}^3} \otimes \mathcal{M}_0 & \longrightarrow & \mathcal{O}_{\mathfrak{X}^2} \otimes \mathcal{M}_0 & \xrightarrow{\epsilon} & \mathcal{O}_{\Delta} \otimes \mathcal{M}_0 \\
& \nearrow & \uparrow & \nearrow & \uparrow & \nearrow & \uparrow & \nearrow & \uparrow \\
\cdots & \longrightarrow & \mathcal{O}_{\mathfrak{X}^4} \otimes \mathcal{M}_1 & \longrightarrow & \mathcal{O}_{\mathfrak{X}^3} \otimes \mathcal{M}_1 & \longrightarrow & \mathcal{O}_{\mathfrak{X}^2} \otimes \mathcal{M}_1 & \xrightarrow{\epsilon} & \mathcal{O}_{\Delta} \otimes \mathcal{M}_1 \\
& \nearrow & \uparrow & \nearrow & \uparrow & \nearrow & \uparrow & \nearrow & \uparrow \\
\cdots & \longrightarrow & \mathcal{O}_{\mathfrak{X}^4} \otimes \mathcal{M}_0 & \longrightarrow & \mathcal{O}_{\mathfrak{X}^3} \otimes \mathcal{M}_0 & \longrightarrow & \mathcal{O}_{\mathfrak{X}^2} \otimes \mathcal{M}_0 & \xrightarrow{\epsilon} & \mathcal{O}_{\Delta} \otimes \mathcal{M}_0 \\
& \nearrow & \uparrow & \nearrow & \uparrow & \nearrow & \uparrow & \nearrow & \uparrow \\
\cdots & \longrightarrow & \mathcal{O}_{\mathfrak{X}^4} \otimes \mathcal{M}_1 & \longrightarrow & \mathcal{O}_{\mathfrak{X}^3} \otimes \mathcal{M}_1 & \longrightarrow & \mathcal{O}_{\mathfrak{X}^2} \otimes \mathcal{M}_1 & \xrightarrow{\epsilon} & \mathcal{O}_{\Delta} \otimes \mathcal{M}_1 \\
& \nearrow & \uparrow & \nearrow & \uparrow & \nearrow & \uparrow & \nearrow & \uparrow \\
& & \vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

where the horizontal maps are induced by b , diagonal maps induced by B_w and the vertical maps by the differential on \mathcal{M} . This "bi-complex" is of course just a mnemonic, but it gives us insight into how to deal with the complex $\mathcal{W} \otimes_{S(\mathcal{O}_{X \times X})} \mathcal{M}$. In particular, we may "filter the bi-complex by rows" to get a filtration on $\mathcal{W} \otimes_{S(\mathcal{O}_{X \times X})} \mathcal{M}$. One should convince oneself that this indeed a filtration by subcomplexes. This filtration is bounded below and exhaustive, therefore the associated spectral sequence converges. Already on the E_1 page all of the groups are 0 since the rows of the "bi-complex" associated to $\mathcal{W} \otimes_{S(\mathcal{O}_{X \times X})} \mathcal{M}$ are exact. This gives us that the map

$$\widehat{\mathcal{B}ar}_{\tilde{w}}(X) \otimes_{S(\mathcal{O}_{X \times X})} \mathcal{M} \rightarrow \Delta_* S(\mathcal{O}_X) \otimes_{S(\mathcal{O}_{X \times X})} \mathcal{M}$$

is a quasi-isomorphism.

Now for general \mathcal{M} , let $\underline{\mathcal{M}} = \text{Tot}(\mathcal{F}^\bullet)$ be a flat replacement of \mathcal{M} , where \mathcal{F}^\bullet is a (finite) resolution of \mathcal{M} by flat $-\tilde{w}$ -curved modules. This can be done as in corollary 3.8; finiteness is possible since X is smooth. It is well known (see for example [16]) that $(p_{1,k})_* \mathcal{O}_{\mathfrak{X}^k}$ is flat as an \mathcal{O}_{X^2} -module and therefore the graded components of $\widehat{\mathcal{B}ar}_{\tilde{w}}$ are flat. This implies the morphism

$$\widehat{\mathcal{B}ar}_{\tilde{w}} \otimes \underline{\mathcal{M}} \rightarrow \widehat{\mathcal{B}ar}_{\tilde{w}} \otimes \mathcal{M}$$

is a quasi-isomorphism.

We have

$$\widehat{\mathcal{B}ar}_{\tilde{w}} \otimes_{S(\mathcal{O}_{X \times X})} \underline{\mathcal{M}} = \widehat{\mathcal{B}ar}_{\tilde{w}} \otimes_{S(\mathcal{O}_{X \times X})} \text{Tot}(\mathcal{F}^\bullet) = \text{Tot}(\widehat{\mathcal{B}ar}_{\tilde{w}} \otimes_{S(\mathcal{O}_{X \times X})} \mathcal{F}^\bullet)$$

The cone of the morphism

$$\text{Tot}(\widehat{\mathcal{B}ar}_{\tilde{w}} \otimes \mathcal{F}^\bullet) \rightarrow \text{Tot}(\Delta_* S(\mathcal{O}_X) \otimes \mathcal{F}^\bullet)$$

is given by $\text{Tot}(\mathcal{W} \otimes \mathcal{F}^\bullet)$, which is the total complex of a bicomplex with exact columns (by the above argument) and uniformly bounded rows and therefore is acyclic.

Therefore we obtain a zig-zag of quasi-isomorphisms

$$\widehat{\mathcal{B}ar}_{\tilde{w}} \otimes_{S(\mathcal{O}_{X \times X})} \mathcal{M} \leftarrow \widehat{\mathcal{B}ar}_{\tilde{w}} \otimes_{S(\mathcal{O}_{X \times X})} \underline{\mathcal{M}} \rightarrow \Delta_*(S(\mathcal{O}_X)) \otimes_{S(\mathcal{O}_{X \times X})} \underline{\mathcal{M}}$$

Since $\Delta_*(S(\mathcal{O}_X)) \otimes_{S(\mathcal{O}_{X \times X})} \underline{\mathcal{M}}$ computes $\Delta_* S(\mathcal{O}_X) \otimes_{S(\mathcal{O}_{X \times X})}^{\mathbb{L}} \mathcal{M}$, we are done. \square

Lemma 4.2. *The map $\widehat{\mathcal{B}ar}_{\tilde{w}} \rightarrow \Delta_* S(\mathcal{O}_X)$ is a weak equivalence in $Z^0 \mathfrak{Qcoh}(X, \mathcal{O}_{X \times X}, \tilde{w})$*

Proof. Again we use \mathcal{W} for the cone of the map $\widehat{\mathcal{B}ar}_{\tilde{w}} \rightarrow \Delta_* S(\mathcal{O}_X)$. Let \mathcal{G} be a compact generator for $\mathfrak{Qcoh}(X \times X, \mathcal{O}_{X \times X}, \tilde{w})$ and by 3.8 we can take \mathcal{G} to be a matrix factorization. By the previous lemma

$$\mathcal{G}^\vee \otimes_{S(\mathcal{O}_{X \times X})} \mathcal{W} = \mathcal{H}om_{S(\mathcal{O}_{X \times X})}(\mathcal{G}, \mathcal{W})$$

is acyclic, and since \mathcal{G} is locally free we have a quasi-isomorphism

$$\mathcal{H}om_{S(\mathcal{O}_{X \times X})}(\mathcal{G}, \mathcal{W}) \cong \mathcal{H}om_{S(\mathcal{O}_{X \times X})}(\mathcal{G}, K(\mathcal{W})),$$

where $K : \mathfrak{Qcoh}(X \times X, \mathcal{O}_{X \times X}, \tilde{w}) \rightarrow \text{Inj}(X \times X, \mathcal{O}_{X \times X}, \tilde{w})$ is our chosen functorial injective replacement. By adjunction, the complex of sheaves $\mathcal{H}om_{S(\mathcal{O}_X)}(\mathcal{G}, K(\mathcal{W}))$ has injective graded components. We are want to say that that having injective graded components is sufficient for $\mathcal{H}om(\mathcal{G}, K(\mathcal{W}))$ to be adapted to the global sections functor, if we could the proof would be done. However this complex is unbounded in both directions so care must be taken.

Since X is smooth thus has finite homological dimension, each of the cokernels of the differentials are injective. Then by exactness, the kernels of the differentials are also injective. Using these facts one can easily verify directly that the global sections functor is exact by checking at any particular spot and truncating appropriately, so that the truncated sequence is a bounded exact sequence of injective sheaves. Finally we can conclude that the complex of vector spaces

$$\text{Hom}(\mathcal{G}, K(\mathcal{W})) = \Gamma(\mathcal{H}om(\mathcal{G}, K(\mathcal{W})))$$

is exact. Since \mathcal{G} is a generator this implies that $K(\mathcal{W})$ is coacyclic and therefore \mathcal{W} is as well. \square

Theorem 4.3. *The Hochschild homology of $\mathfrak{MF}(X, \mathcal{O}_X, w)$ is $\mathbb{R}\Gamma(\Omega_{dw})$, where Ω_{dw} is the two periodic complex of sheaves*

$$\dots \xrightarrow{dw \wedge} \bigoplus_{i \text{ odd}} \Omega^i \xrightarrow{dw \wedge} \bigoplus_{i \text{ even}} \Omega^i \xrightarrow{dw \wedge} \bigoplus_{i \text{ odd}} \Omega^i \xrightarrow{dw \wedge} \bigoplus_{i \text{ even}} \Omega^i \xrightarrow{dw \wedge} \dots$$

with $\bigoplus_{i \text{ even}} \Omega^i$ in even degrees.

Proof. By lemmas 3.12 and 3.11 we compute the hochschild homology of $\mathfrak{MF}(X, \mathcal{O}_X, w)$ as $\mathbb{R}\Gamma(\mathbb{L}\Delta^* \Delta_* S(\mathcal{O}_X))$. By lemma 4.1, we may compute $\mathbb{L}\Delta^* \Delta_* S(\mathcal{O}_X)$ as $\Delta^* \widehat{\mathcal{B}ar}_{\tilde{w}}$. Now,

$\Delta^* \widehat{\mathcal{B}ar}_{\tilde{w}}$ is given as the total complex of the “bi-complex”

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \nwarrow B_w & & \nwarrow B_w & & \nwarrow B_w \\
 \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & \nearrow b & \Delta^* \mathcal{O}_{\mathfrak{x}^4} & \xrightarrow{b} & \Delta^* \mathcal{O}_{\mathfrak{x}^3} & \xrightarrow{b} & \Delta^* \mathcal{O}_{\mathfrak{x}^2} \\
 & & \nwarrow B_w & & \nwarrow B_w & & \nwarrow B_w \\
 \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & \nearrow b & \Delta^* \mathcal{O}_{\mathfrak{x}^4} & \xrightarrow{b} & \Delta^* \mathcal{O}_{\mathfrak{x}^3} & \xrightarrow{b} & \Delta^* \mathcal{O}_{\mathfrak{x}^2} \\
 & & \nwarrow B_w & & \nwarrow B_w & & \nwarrow B_w \\
 \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \Delta^* \mathcal{O}_{\mathfrak{x}^4} & & \Delta^* \mathcal{O}_{\mathfrak{x}^3} & & \Delta^* \mathcal{O}_{\mathfrak{x}^2} \\
 & & \vdots & & \vdots & & \vdots
 \end{array}$$

(2)

Applying the Hochschild-Kostant-Rosenburg (HKR) quasi-isomorphism, which is given locally by

$$a_0 \boxtimes \cdots \boxtimes a_q \mapsto \frac{1}{q!} a_0 a_q da_1 \wedge \cdots \wedge da_{q-1}$$

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(see [16] theorem 4.8) along the rows we obtain a quasi-isomorphism between (2) and the bicomplex

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & \swarrow \wedge dw & & \swarrow \wedge dw & & \swarrow \wedge dw & \\
 \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & \swarrow \wedge dw & & \swarrow \wedge dw & & \swarrow \wedge dw & \\
 \dots & \xrightarrow{0} & \Omega^2 & \xrightarrow{0} & \Omega & \xrightarrow{0} & \mathcal{O}_X \\
 & \swarrow \wedge dw & & \swarrow \wedge dw & & \swarrow \wedge dw & \\
 \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & \swarrow \wedge dw & & \swarrow \wedge dw & & \swarrow \wedge dw & \\
 \dots & \xrightarrow{0} & \Omega^2 & \xrightarrow{0} & \Omega & \xrightarrow{0} & \mathcal{O}_X \\
 & \swarrow \wedge dw & & \swarrow \wedge dw & & \swarrow \wedge dw & \\
 \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & \swarrow \wedge dw & & \swarrow \wedge dw & & \swarrow \wedge dw & \\
 \dots & \xrightarrow{0} & \Omega^2 & \xrightarrow{0} & \Omega & \xrightarrow{0} & \mathcal{O}_X \\
 & & \vdots & & \vdots & & \vdots
 \end{array}$$

(3)

Under the HKR quasi-isomorphism the map B_w does indeed become $\wedge dw$: locally we have

$$dw \wedge HKR((1 \boxtimes a_1 \boxtimes a_2 \cdots \boxtimes a_q \boxtimes 1) \boxtimes 1) = \frac{1}{q!} dw \wedge da_1 \wedge \cdots \wedge da_q$$

and

$$\begin{aligned}
 HKR(B_w(1 \boxtimes a_1 \boxtimes \cdots \boxtimes a_q \boxtimes 1)) &= \frac{1}{q+1!} \sum_{i=0}^{k+1} (-1)^i da_1 \wedge \cdots \wedge da_i \wedge dw \wedge da_{i+1} \wedge \cdots \wedge da_q \\
 &= \frac{1}{q!} dw \wedge da_1 \wedge \cdots \wedge da_q.
 \end{aligned}$$

This gives that the Hochschild homology of $\mathfrak{MF}_{loc}(X, \mathcal{O}_X, w)$ is given as the hypercohomology of the complex

$$\dots \xrightarrow{\wedge dw} \bigoplus_{i \text{ even}} \Omega^i \xrightarrow{dw \wedge} \bigoplus_{i \text{ odd}} \Omega^i \xrightarrow{dw \wedge} \bigoplus_{i \text{ even}} \Omega^i \xrightarrow{dw \wedge} \bigoplus_{i \text{ odd}} \Omega^i \xrightarrow{dw \wedge} \dots$$

where $\bigoplus_{i \text{ even}} \Omega^i$ is in even degrees and $dw \wedge$ wedges dw in the first slot. \square

5. BOUNDARY-BULK AND CHERN CHARACTER

Let us recall from [6] the definition of the *boundary-bulk map*

$$\tau_{\mathcal{E}} : \text{Hom}_{\mathfrak{MF}_{loc}(w)}(\mathcal{E}, \mathcal{E}) \rightarrow \text{HH}(\mathfrak{MF}_{loc}(X, \mathcal{L}, w)).$$

We have a natural quasi-isomorphism

$$\mathrm{Hom}_{\mathfrak{M}\mathfrak{F}_{loc}(w)}(\mathcal{E}, \mathcal{E}) \cong \mathrm{Tr}(\mathcal{E} \boxtimes \mathcal{E}^\vee)$$

Then we apply the trace functor the evaluation map $\mathcal{E} \boxtimes \mathcal{E}^\vee \rightarrow \Delta_* S(\mathcal{O}_X)$ to get a map

$$\tau_{\mathcal{E}} : \mathrm{Hom}_{\mathfrak{M}\mathfrak{F}_{loc}(X, \mathcal{L}, w)}(\mathcal{E}, \mathcal{E}) \cong \mathrm{Tr}(\mathcal{E} \boxtimes \mathcal{E}^\vee) \rightarrow \mathrm{Tr}(\Delta_* S(\mathcal{O}_X)) = \mathrm{HH}(\mathfrak{M}\mathfrak{F}_{loc}(X, \mathcal{L}, w)).$$

Of course, this construction works for any DG-category, we refer the reader to *loc. cit.* for the details and also for the proof of the fact that the Chern Character in the sense of [14] is $\tau_{\mathcal{E}}(id)$.

Now, having computed the Hochschild homology for the category $\mathfrak{M}\mathfrak{F}_{loc}(X, \mathcal{O}_X, w)$ as $\mathbb{R}\Gamma(\Omega_{dw})$ and now making the trivial observation that since \mathcal{E} is locally free we have

$$\mathbb{R}\mathrm{Hom}(\mathcal{E}, \mathcal{F}) = \mathbb{R}\Gamma\mathrm{Hom}(\mathcal{E}, \mathcal{F}),$$

one is want to promote the boundary bulk-map to a map in the derived category of sheaves on X :

$$\mathcal{T}_{\mathcal{E}} : \mathcal{H}om(\mathcal{E}, \mathcal{E}) \rightarrow \Omega_{dw},$$

and thereby understand the particular invariants we wish to compute in two steps, first to get an explicit representative for $\mathcal{T}_{\mathcal{E}}$ and then to understand the more classical problem of deducing the induced map on cohomology.

Lemma 5.1. *Define a map $\mathcal{T}_{\mathcal{E}} : \mathcal{H}om(\mathcal{E}, \mathcal{E}) \rightarrow \Omega_{dw}$ in $D(X)$ by*

$$\mathcal{H}om(\mathcal{E}, \mathcal{E}) = \mathcal{E} \otimes \mathcal{E}^\vee \cong \mathbb{L}\Delta^*(\mathcal{E} \boxtimes \mathcal{E}^\vee) \xrightarrow{eval} \mathbb{L}\Delta^*(\Delta_* S(\mathcal{O}_X)) \cong \Omega_{dw}$$

Then $\tau_{\mathcal{E}} = \mathbb{R}\Gamma(\mathcal{T}_{\mathcal{E}})$.

Proof. This is clear. □

We wish now to get a better handle on this map $\mathcal{T}_{\mathcal{E}}$. We may resolve a matrix factorization \mathcal{E} by $\epsilon \otimes 1 : \widehat{\mathcal{B}ar}_{\tilde{w}} \otimes_{S(\mathcal{O}_X)} \mathcal{E} \rightarrow \Delta_* S(\mathcal{O}_X) \otimes_{S(\mathcal{O}_X)} \mathcal{E} = \mathcal{E}$. Here we use the short hand $\otimes_{S(\mathcal{O}_X)}$ between an \tilde{w} curved module on $X \times X$ and a w -curved module on X to mean

$$\widehat{\mathcal{B}ar}_{\tilde{w}} \otimes_{S(\mathcal{O}_X)} \mathcal{E} := (p_1)_*(\widehat{\mathcal{B}ar}_{\tilde{w}} \otimes_{S(\mathcal{O}_{X^2})} p_2^* \mathcal{E})$$

where p_1 and p_2 are the natural projections from $X \times X$ to X . Since matrix factorizations are flat, lemma 4.1 implies that this map is a weak equivalence in $Z^0 \mathbf{Qcoh}(w)$. Then the map

$$\mathcal{E}^\vee \otimes (\widehat{\mathcal{B}ar}_{\tilde{w}} \otimes_{S(\mathcal{O}_X)} \mathcal{E}) \rightarrow \mathcal{E}^\vee \otimes \mathcal{E} = \mathcal{H}om(\mathcal{E}, \mathcal{E})$$

is a quasi-isomorphism of complexes of sheaves.

This gives us an explicit representative for $\mathcal{T}_{\mathcal{E}}$ given by the roof

$$\begin{array}{ccc} \mathcal{E}^\vee \otimes (\widehat{\mathcal{B}ar}_{\tilde{w}} \otimes_{S(\mathcal{O}_X)} \mathcal{E}) & \xrightarrow{1 \otimes \sigma} & \mathcal{E}^\vee \otimes \mathcal{E} \otimes_{S(\mathcal{O}_X)} \widehat{\mathcal{B}ar}_{\tilde{w}} \\ \downarrow 1 \otimes \epsilon \otimes 1 \wr & & \downarrow ev \otimes 1 \otimes 1 \\ \mathcal{E}^\vee \otimes \mathcal{E} & & \Delta^* \widehat{\mathcal{B}ar}_{\tilde{w}} \\ & & \downarrow HKR \\ & & \Omega_{dw} \end{array}$$

where ev is the evaluation map of \mathcal{E}^\vee on \mathcal{E} , σ is switching the factors in the tensor product and $\overline{\otimes}$ is contraction of tensor.

The goal now is to construct a natural morphism

$$\mathcal{E}xp(at(\mathcal{E})) : \mathcal{E} \rightarrow \Omega_{dw} \underset{S(\mathcal{O}_X)}{\otimes} \mathcal{E}$$

in the coderived category of w -curved modules, such that

$$\mathcal{T}_{\mathcal{E}} = str(- \circ \mathcal{E}xp(at(\mathcal{E}))) : \mathcal{H}om(\mathcal{E}, \mathcal{E}) \rightarrow \mathcal{H}om(\mathcal{E}, \Omega_{dw} \underset{S(\mathcal{O}_X)}{\otimes} \mathcal{E}) = \mathcal{H}om(\mathcal{E}, \mathcal{E}) \underset{S(\mathcal{O}_X)}{\otimes} \Omega_{dw} \rightarrow \Omega_{dw}$$

This morphism will then be a sort of internal Chern Character for the category of matrix factorization. In what follows we will want to fix $n = \dim(X)$.

Before we proceed we wish take a motivational digression and consider the category of complexes of coherent sheaves on X . We will follow very closely the treatment from [3]. The idea is that in *loc. cit* Markarian constructs an internal Chern Character by exponentiating the Atiyah class map and which takes values in Hochschild homology sheaves. We wish to mimic this construction. The main technical problem, as we will see, is that there is no obvious analog to the Atiyah class for matrix factorizations. But, oddly enough, even though the class $at(\mathcal{E})$ does not seem to exist, its exponential does.

We have the exact sequence of \mathcal{O}_{X^2} modules

$$0 \rightarrow \mathcal{I}/\mathcal{I}^2 \rightarrow \mathcal{O}_{X^2}/\mathcal{I}^2 \rightarrow \mathcal{O}_{\Delta} \rightarrow 0$$

where \mathcal{I} is the kernel of the multiplication map $\mathcal{O}_{X^2} \rightarrow \mathcal{O}_{\Delta}$. We will write Ω_{Δ} for $\mathcal{I}/\mathcal{I}^2$ and \mathcal{J}_{Δ}^1 for $\mathcal{O}_{X^2}/\mathcal{I}^2$. Given an honest complex ($d^2 = 0$) of sheaves, \mathcal{E} , we may “tensor on the right” by \mathcal{E} to get an exact sequence of \mathcal{O}_X -complexes

$$(4) \quad 0 \rightarrow \Omega^1 \underset{\mathcal{O}_X}{\otimes} \mathcal{E} \rightarrow \mathcal{J}^1 \otimes_{\mathcal{O}_X} \mathcal{E} \rightarrow \mathcal{E} \rightarrow 0.$$

where for an \mathcal{O}_{X^2} module \mathcal{M} and an \mathcal{O}_X -module \mathcal{F}

$$\mathcal{M} \underset{\mathcal{O}_X}{\otimes} \mathcal{F} := (p_1)_*(\mathcal{M} \underset{\mathcal{O}_{X^2}}{\otimes} p_2^*(\mathcal{F}))$$

where $p_i : X \times X \rightarrow X$ are the standard projections. The extension in (4) gives an element of

$$Ext^1(\mathcal{E}, \Omega^1 \underset{\mathcal{O}_X}{\otimes} \mathcal{E}) = Hom_{D(X)}(\mathcal{E}, \Omega^1 \underset{\mathcal{O}_X}{\otimes} \mathcal{E}[1]).$$

This element, $at(\mathcal{E}) : \mathcal{E} \rightarrow \Omega \underset{\mathcal{O}_X}{\otimes} \mathcal{E}[1]$, is called the Atiyah class of \mathcal{E} .

Composing the morphism $at(\mathcal{E})$ with itself i times and then wedging forms we obtain a map

$$\wedge at(\mathcal{E})^i : \mathcal{E} \rightarrow \Omega^i \underset{\mathcal{O}_X}{\otimes} \mathcal{E}[i].$$

Using the isomorphism $\mathcal{O}_{\Delta} \underset{\mathcal{O}_X}{\otimes} \Omega^1 \cong \Omega_{\Delta}$, get a long exact sequence

$$0 \rightarrow \Omega^{\otimes i-1} \underset{\mathcal{O}_X}{\otimes} \Omega_{\Delta} \rightarrow \Omega^{\otimes i-1} \underset{\mathcal{O}_X}{\otimes} \mathcal{J}_{\Delta}^1 \rightarrow \cdots \rightarrow \Omega^1 \underset{\mathcal{O}_X}{\otimes} \mathcal{J}_{\Delta}^1 \rightarrow \mathcal{J}_{\Delta}^1 \rightarrow \mathcal{O}_{\Delta}$$

Tensoring this sequence on the right with \mathcal{E} we get a long exact sequence

$$(5) \quad 0 \rightarrow \Omega^{\otimes i} \underset{\mathcal{O}_X}{\otimes} \mathcal{E} \rightarrow \Omega^{\otimes i-1} \underset{\mathcal{O}_X}{\otimes} \mathcal{J}^1(\mathcal{E}) \rightarrow \cdots \rightarrow \Omega^1 \underset{\mathcal{O}_X}{\otimes} \mathcal{J}^1(\mathcal{E}) \rightarrow \mathcal{J}^1(\mathcal{E}) \rightarrow \mathcal{E}.$$

Here we denote by $\mathcal{J}^1(\mathcal{E})$ the tensor product $\mathcal{J}_\Delta^1 \otimes_{\mathcal{O}_X} \mathcal{E}$. One sees easily that this exact sequence represents $\wedge \text{at}(\mathcal{E})^i$ as a Yoneda extension and so the map $\wedge \text{at}(\mathcal{E})^i$ is given as the zig-zag

$$\mathcal{E} \leftarrow (\Omega^{\otimes i} \otimes_{\mathcal{O}_X} \mathcal{E} \rightarrow \Omega^{\otimes i-1} \otimes_{\mathcal{O}_X} \mathcal{J}^1(\mathcal{E}) \rightarrow \cdots \rightarrow \Omega^1 \otimes_{\mathcal{O}_X} \mathcal{J}^1(\mathcal{E}) \rightarrow \mathcal{J}^1(\mathcal{E})) \rightarrow \Omega^i \otimes \mathcal{E}[i]$$

where the last map is simply projection onto the last factor followed by wedging forms.

When we try to mimic this construction for curved ($w \neq 0$) modules, the projection onto the last factor is no longer a map in the category we care about. Or more accurately the inclusion of graded \mathcal{O}_X modules $\Omega^i \otimes \mathcal{E}[i] \rightarrow \Omega_{dw} \otimes \mathcal{E}[i]$ is not a map of curved modules, unless $i = n$ or $dw = 0$. Our first observation is that we can view the exponential of the Atiyah class as a map from the total complex of the resolution,

$$\Omega^{\otimes n} \otimes_{\mathcal{O}_X} \mathcal{E} \rightarrow \Omega^{\otimes n-1} \otimes_{\mathcal{O}_X} \mathcal{J}^1(\mathcal{E}) \rightarrow \cdots \rightarrow \Omega^1 \otimes_{\mathcal{O}_X} \mathcal{J}^1(\mathcal{E}) \rightarrow \mathcal{J}^1(\mathcal{E})$$

of \mathcal{E} , to $\Omega^\bullet \otimes \mathcal{E}$, by using the various projections onto $\Omega^{\otimes i} \otimes \mathcal{E}$, for $i \leq n$, where again $n = \dim(X)$. The second observation is that we still can in $\mathfrak{M}\mathfrak{S}_{loc}(X, \mathcal{L}, w)$ construct appropriate analogs of this resolution of \mathcal{E} . We do this now.

As with the curved bar complex we may use the resolution

$$(6) \quad 0 \rightarrow \Omega^{\otimes n-1} \otimes_{\mathcal{O}_X} \Omega_\Delta \rightarrow \Omega^{\otimes n-1} \otimes_{\mathcal{O}_X} \mathcal{J}_\Delta^1 \rightarrow \cdots \rightarrow \Omega^1 \otimes_{\mathcal{O}_X} \mathcal{J}_\Delta^1 \rightarrow \mathcal{J}_\Delta^1 \rightarrow \mathcal{O}_\Delta$$

to build a \tilde{w} curved complex $\mathcal{A}t$ which resolves $\Delta_* S(\mathcal{O}_X)$. Set

$$\mathcal{A}_i = \begin{cases} \Omega^{\otimes i} \otimes_{\mathcal{O}_X} \mathcal{J}_\Delta^1 & \text{if } 0 \leq i < n \\ \Omega^{\otimes(n-1)} \otimes_{\mathcal{O}_X} \Omega_\Delta & \text{if } i = n \\ 0 & \text{else} \end{cases}$$

Then define the graded components of $\mathcal{A}t$ by folding 2-periodically:

$$\mathcal{A}t_i = \bigoplus_{j \equiv i \pmod{2}} \mathcal{A}_j.$$

We have the differential, $m : \mathcal{A}t_i \rightarrow \mathcal{A}t_{i+1}$ coming from the resolution 6 which (locally) is given by the equation

$$\begin{aligned} m(da_1 \otimes da_2 \otimes \cdots \otimes da_n \otimes a_0 \boxtimes a_{n+1}) &= a_0 a_{n+1} da_1 \otimes \cdots \otimes da_{n-1} \otimes a_n \boxtimes 1 \\ &\quad - a_0 a_{n+1} da_1 \otimes \cdots \otimes da_{n-1} \otimes 1 \boxtimes a_{n-1}. \end{aligned}$$

Here we have chosen indices in preparation for certain morphisms involving the curved bar complex. Again we use \boxtimes to emphasize external tensor. Depending on our purposes, i.e. whether we want to emphasize or deemphasize the role of \mathcal{J}_Δ^1 in the tensor $\Omega^{\otimes q} \otimes \mathcal{J}_\Delta^1$, we will alternatively simply write

$$a_0 da_1 \otimes \cdots \otimes da_q \boxtimes a_{q+1} = a_1 \otimes \cdots \otimes a_q \otimes a_0 \boxtimes a_{q+1}$$

Coordinate free, this map m is simply induced by the multiplication map $\mathcal{J}_\Delta^1 \rightarrow \mathcal{O}_\Delta$ followed by the isomorphism $\Omega^{\otimes i} \otimes_{\mathcal{O}_X} \mathcal{O}_\Delta \cong \Omega^{\otimes(i-1)} \otimes_{\mathcal{O}_X} \Omega_\Delta$ and then the inclusion

$$\Omega^{\otimes(i-1)} \otimes_{\mathcal{O}_X} \Omega_\Delta \rightarrow \Omega^{\otimes(i-1)} \otimes_{\mathcal{O}_X} \mathcal{J}_\Delta^1.$$

And, of course, on the summand $\mathcal{A}_n = \Omega^{\otimes(n-1)} \otimes_{\mathcal{O}_X} \Omega_\Delta$, m is simply the inclusion of $\Omega^{\otimes(n-1)} \otimes_{\mathcal{O}_X} \Omega_\Delta$ into $\Omega^{\otimes(n-1)} \otimes_{\mathcal{O}_X} \mathcal{J}_\Delta^1$. To curve $\mathcal{A}t$ by \tilde{w} we add a second differential B_{dw} given by the formula

$$B_{dw}(\omega_1 \otimes \cdots \otimes \omega_n \otimes a_0 \boxtimes a_{q+1}) = \sum_{i=0}^q (-1)^i \omega_1 \otimes \cdots \otimes \omega_i \otimes dw \otimes \omega_{i+1} \otimes \cdots \otimes \omega_q \otimes a_0 \boxtimes a_{q+1}.$$

As with $\widehat{\mathcal{B}ar}_{\tilde{w}}$, we may picture $\mathcal{A}t$ as the total complex of the bicomplex:

$$\begin{array}{ccccccc}
& & \vdots & & \vdots & & \vdots \\
& & \swarrow B_{dw} & & \swarrow B_{dw} & & \swarrow B_{dw} \\
0 & \cdots & \cdots & \cdots & 0 & \cdots & 0 \\
& & \swarrow B_{dw} & & \swarrow B_{dw} & & \swarrow B_{dw} \\
\Omega^{\otimes n} \otimes \mathcal{J}_{\Delta}^1 & \xrightarrow{b} & \cdots & \xrightarrow{b} & \Omega^{\otimes 2} \otimes \mathcal{J}_{\Delta}^1 & \xrightarrow{b} & \Omega \otimes \mathcal{J}_{\Delta}^1 \xrightarrow{b} \mathcal{J}_{\Delta}^1 \\
& & \swarrow B_{dw} & & \swarrow B_{dw} & & \swarrow B_{dw} \\
0 & \cdots & \cdots & \cdots & 0 & \cdots & 0 \\
& & \swarrow B_{dw} & & \swarrow B_{dw} & & \swarrow B_{dw} \\
\Omega^{\otimes n} \otimes \mathcal{J}_{\Delta}^1 & \xrightarrow{b} & \cdots & \xrightarrow{b} & \Omega^{\otimes 2} \otimes \mathcal{J}_{\Delta}^1 & \xrightarrow{b} & \Omega \otimes \mathcal{J}_{\Delta}^1 \xrightarrow{b} \mathcal{J}_{\Delta}^1 \\
& & \swarrow B_{dw} & & \swarrow B_{dw} & & \swarrow B_{dw} \\
0 & \cdots & \cdots & \cdots & 0 & \cdots & 0 \\
& & \vdots & & \vdots & & \vdots
\end{array}$$

Now we claim that $\mathcal{A}t$ imbued with the differential $B_{dw} - m\gamma$ is a \tilde{w} curved module, where γ is the grading operator with respect to forms, i.e. $\gamma|_{\Omega^q \otimes \mathcal{J}_{\Delta}^1} = (-1)^q$. Indeed the computations

$$\begin{aligned}
& m\gamma B_{dw}(a_0 da_1 \otimes \cdots \otimes da_q \boxtimes a_{q+1}) \\
&= (-1)^{q+1} \left[\sum_{i=0}^q (-1)^i m(a_0 da_1 \otimes \cdots \otimes dw \otimes \cdots \otimes da_q \boxtimes a_{q+1}) \right] \\
&= (-1)^{q+1} \left[(-1)^q (a_0 a_{q+1} da_1 \otimes \cdots \otimes da_q \otimes w \boxtimes 1 - a_0 a_{q+1} da_1 \otimes \cdots \otimes da_q \otimes 1 \boxtimes w) \right. \\
&\quad + \sum_{i=0}^{q-1} (-1)^i a_0 a_{q+1} da_1 \otimes \cdots \otimes da_i \otimes dw \otimes da_{i+1} \otimes \cdots \otimes da_{q-1} \otimes a_q \boxtimes 1 \\
&\quad \left. - \sum_{i=0}^{q-1} (-1)^i a_0 a_{q+1} da_1 \otimes \cdots \otimes da_i \otimes dw \otimes da_{i+1} \otimes \cdots \otimes da_{q-1} \otimes 1 \boxtimes a_q \right]
\end{aligned}$$

and

$$\begin{aligned}
& B_{dw} m\gamma(da_1 \otimes \cdots \otimes da_q \otimes a_0 \boxtimes a_{q+1}) \\
&= (-1)^q \left[\sum_{i=0}^{q-1} (-1)^i a_0 a_{q+1} da_1 \otimes \cdots \otimes da_i \otimes dw \otimes da_{i+1} \otimes \cdots \otimes da_{q-1} \otimes a_q \boxtimes 1 \right. \\
&\quad \left. - \sum_{i=0}^{q-1} (-1)^i a_0 a_{q+1} da_1 \otimes \cdots \otimes da_i \otimes dw \otimes da_{i+1} \otimes \cdots \otimes da_{q-1} \otimes 1 \boxtimes a_q \right]
\end{aligned}$$

show that

$$\begin{aligned}
(B_{dw} - m\gamma)^2 &= -B_{dw}m\gamma - m\gamma B_{dw}(da_1 \otimes \cdots \otimes da_q \otimes a_0 \boxtimes a_{q+1}) \\
&= a_0 a_{q+1} da_1 \otimes \cdots \otimes da_q \otimes (w \boxtimes 1 - 1 \boxtimes w) \\
&= da_1 \otimes \cdots \otimes da_q \otimes (a_0 a_{q+1} w \boxtimes 1 - a_0 a_{q+1} \boxtimes w)
\end{aligned}$$

The final observations are that \tilde{w} acts on $\Omega^{\otimes q} \otimes_{\mathcal{O}_X} \mathcal{J}_\Delta^1$ by

$$\tilde{w} \cdot \omega_1 \otimes \cdots \omega_q \otimes a_0 \boxtimes a_{q+1} = w\omega_1 \otimes \cdots \omega_q \otimes a_0 \boxtimes a_{q+1}w = \omega_1 \otimes \cdots \omega_q \otimes wa_0 \boxtimes a_{q+1}w$$

and the difference between this action and the above computation for $(B_{dx} - m\gamma)^2$ is

$$a_0 a_{q+1} w \boxtimes 1 - a_0 a_{q+1} \boxtimes w - wa_0 \boxtimes a_{q+1} + a_0 \boxtimes wq = (a_0 \boxtimes 1)(w \boxtimes 1 - 1 \boxtimes w)(a_{q+1} \boxtimes 1 - 1 \boxtimes a_{q+1})$$

which is 0 in \mathcal{J}_Δ^1 . Therefore the map $(B_{dw} - m\gamma)^2 = -B_{dw}m\gamma - m\gamma B_{dw}$ is indeed multiplication by \tilde{w} .

Now there are maps $\pi : \mathcal{O}_{\mathfrak{X}^{q+2}} \rightarrow \Omega^{\otimes q} \otimes_{\mathcal{O}_X} \mathcal{J}_\Delta^1$ given by

$$\pi(a_0 \boxtimes a_1 \boxtimes \cdots \boxtimes a_q \boxtimes a_{q+1}) = a_0 da_1 \otimes \cdots \otimes da_q \boxtimes a_{q+1}$$

It is easy to see $\pi B_w = B_{dw}\pi$. We observe that for an elements of $\mathcal{O}_{\mathfrak{X}^{q+1}}$ of the form $a_0 \boxtimes \cdots \boxtimes a_i a_{i+1} \boxtimes \cdots \boxtimes a_{q+1}$, with $0 < i < q+1$ we have

$$\begin{aligned}
\pi(a_0 \boxtimes \cdots \boxtimes a_i a_{i+1} \boxtimes \cdots \boxtimes a_{q+1}) &= da_1 \otimes \cdots \otimes d(a_i a_{i+1}) \otimes da_q \otimes a_0 \boxtimes a_{q+1} \\
&= a_0 a_{i+1} da_1 \otimes \cdots \otimes da_i \otimes da_{i+2} \otimes \cdots \otimes da_q \boxtimes a_{q+1} \\
&\quad + a_0 a_i da_1 \otimes \cdots \otimes da_{i-1} \otimes da_{i+1} \otimes \cdots \otimes da_q \boxtimes a_{q+1} \\
&= a_0 a_i da_1 \otimes \cdots \otimes \widehat{da_i} \otimes \cdots \otimes da_q \boxtimes a_{q+1} \\
&\quad + a_0 a_i da_1 \otimes \cdots \otimes \widehat{da_{i+1}} \otimes \cdots \otimes da_q \boxtimes a_{q+1}
\end{aligned}$$

where $\widehat{da_i}$ indicates to omit this tensor. We also have

$$\begin{aligned}
\pi(a_0 a_1 \boxtimes a_2 \boxtimes \cdots \boxtimes a_{q+1}) &= a_0 a_1 \otimes a_2 \otimes \cdots \otimes da_q \boxtimes a_{q+1} \\
&= a_0 a_1 \otimes \widehat{da_1} \otimes da_2 \otimes \cdots \otimes da_q \boxtimes a_{q+1}
\end{aligned}$$

So, again using b for the Hochschild differential, we have

$$\begin{aligned}
\pi b(a_0 \boxtimes \cdots \boxtimes a_{q+1}) &= \sum_{i=0}^q (-1)^i \pi(a_0 \boxtimes \cdots \boxtimes a_i a_{i+1} \boxtimes \cdots \boxtimes a_{q+1}) \\
&= (-1)^q a_0 da_1 \otimes \cdots \otimes da_{q-1} \boxtimes a_q a_{q+1} + a_0 a_1 da_2 \otimes \cdots \otimes da_n \boxtimes a_{n+1} \\
&\quad + \sum_{i=1}^{q-1} (-1)^i a_0 a_i \otimes da_1 \otimes \cdots \otimes \widehat{da_i} \otimes \cdots \otimes da_q \boxtimes da_{q+1} \\
&\quad + \sum_{i=1}^{q-1} (-1)^i a_0 a_{i+1} \otimes da_1 \otimes \cdots \otimes \widehat{da_{i+1}} \otimes \cdots \otimes da_q \boxtimes da_{q+1} \\
&= (-1)^q a_0 da_1 \otimes \cdots \otimes da_{q-1} \boxtimes a_q a_{q+1} + a_0 a_1 da_2 \otimes \cdots \otimes da_n \boxtimes a_{n+1} \\
&\quad + \sum_{i=1}^{q-1} (-1)^i a_0 a_i \otimes da_1 \otimes \cdots \otimes \widehat{da_i} \otimes \cdots \otimes da_q \boxtimes da_{q+1} \\
&\quad + \sum_{i=2}^q (-1)^{i-1} a_0 a_i \otimes da_1 \otimes \cdots \otimes \widehat{da_i} \otimes \cdots \otimes da_q \boxtimes da_{q+1} \\
&= (-1)^q a_0 da_1 \otimes \cdots \otimes da_{q-1} \boxtimes a_q a_{q+1} + a_0 a_1 da_2 \otimes \cdots \otimes da_q \boxtimes a_{q+1} \\
&\quad - a_0 a_1 da_2 \otimes \cdots \otimes \widehat{da_i} \otimes \cdots \otimes da_q \boxtimes da_{q+1} \\
&\quad + (-1)^{q-1} a_0 da_q da_1 \otimes \cdots \otimes da_{q-1} \boxtimes a_{q+1} \\
&= (-1)^{q+1} m \pi(a_0 \boxtimes \cdots \boxtimes a_{q+1})
\end{aligned}$$

The above discussion proves the following lemma:

Lemma 5.2. *The map $\pi : \widehat{\mathcal{B}ar}_{\tilde{w}} \rightarrow \mathcal{A}t$ is a closed morphism of \tilde{w} -curved modules.*

Incidentally this discussion also explains the appearance of the grading operator in the horizontal direction.

Remark 5.3. It is clear that $\pi : \widehat{\mathcal{B}ar}_{\tilde{w}} \rightarrow \mathcal{A}t$ is a weak equivalence of \tilde{w} curved modules on $X \times X$, since both $\widehat{\mathcal{B}ar}_{\tilde{w}}$ and $\mathcal{A}t$ are weakly equivalent to $\Delta_* S(\mathcal{O}_X)$ via projection.

As a piece of notation, for $\mathcal{E} \in \mathfrak{Qcoh}(X, \mathcal{O}_X, w)$, we define

$$\mathcal{A}t(\mathcal{E}) := \mathcal{A}t \otimes_{S(\mathcal{O}_X)} \mathcal{E} := (p_1)_* (\mathcal{A}t \otimes_{S(\mathcal{O}_{X^2})} p_2^* \mathcal{E}).$$

Lemma 5.4. *Let $\wedge : \Omega^{\otimes q} \otimes \mathcal{J}^1(\mathcal{E}) \rightarrow \Omega^q \otimes \mathcal{E}$ denote the anti-symmetrization map:*

$$\wedge(a_0 da_1 \otimes \cdots \otimes da_q \boxtimes e) = a_0 da_1 \wedge \cdots \wedge da_q \otimes e$$

Then the map

$$\sum_{i=0}^n \frac{\wedge}{i!} : \mathcal{A}t(\mathcal{E}) \rightarrow \Omega_{dw} \otimes_{S(\mathcal{O}_X)} \mathcal{E}$$

gives a closed degree 0 morphism of w -curved modules.

Proof. This follows from the calculations

$$\begin{aligned}
& \frac{\wedge}{(q+1)!} B_{dw}(a_0 da_1 \otimes \dots \otimes da_q \boxtimes e) \\
&= \frac{1}{(q+1)!} \sum_{i=0}^q (-1)^i a_0 da_1 \wedge \dots \wedge da_i \wedge dw \wedge da_{i+1} \wedge \dots \wedge da_q \otimes e \\
&= \frac{1}{(q+1)!} \sum_{i=0}^q a_0 dw \wedge da_1 \wedge \dots \wedge da_q \otimes e \\
&= \frac{1}{q!} dw \wedge a_0 da_1 \wedge \dots \wedge da_q \otimes e \\
&= dw \wedge \left(\frac{\wedge}{q!} (a_0 a_1 \otimes \dots \otimes da_n \boxtimes e) \right)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\wedge}{(q-1)!} m(a_0 da_1 \otimes \dots \otimes da_q \boxtimes e) &= \frac{1}{(q-1)!} (a_0 a_q da_1 \wedge \dots \wedge da_{q-1} \otimes e) \\
&\quad - \frac{1}{(q-1)!} (a_0 da_1 \wedge \dots \wedge da_{q-1} \otimes a_q e) \\
&= 0
\end{aligned}$$

and the observation that the differential on \mathcal{E} obviously commutes with the map $\sum_i \wedge_i$. \square

Definition 5.5. Define the map $\mathcal{E}xp(at(\mathcal{E})) : \mathcal{E} \rightarrow \Omega_{dw} \otimes \mathcal{E}$ in the category $D^{co}\mathfrak{Qcoh}(X, \mathcal{O}_X, w)$ by the roof

$$\begin{array}{ccc}
& \mathcal{A}t(\mathcal{E}) & \\
\swarrow \scriptstyle \sim & & \searrow \scriptstyle \Sigma \frac{\wedge}{i!} \\
\mathcal{E} & & \Omega_{dw} \otimes \mathcal{E}
\end{array}$$

Lemma 5.6. The sheafified boundary bulk map $\mathcal{T}_{\mathcal{E}} : \mathcal{H}om_{\mathfrak{S}(\mathcal{O}_X)}(\mathcal{E}, \mathcal{E}) \rightarrow \Omega_{dw}$ is given by $str(- \circ \mathcal{E}xp(at(\mathcal{E})))$, where $str : \mathcal{H}om_{S(\mathcal{O}_X)}(\mathcal{E}, \mathcal{E}) \rightarrow S(\mathcal{O}_X)$ is the super-trace map.

Proof. Recall from the discussion at the beginning of this section that we have the following representative for $\mathcal{T}_{\mathcal{E}}$

$$\begin{array}{ccc}
\mathcal{E}^\vee \otimes (\hat{B}_{\bar{w}} \otimes_{S(\mathcal{O}_X)} \mathcal{E}) & \xrightarrow{1 \otimes \sigma} & \mathcal{E}^\vee \otimes \mathcal{E} \otimes_{S(\mathcal{O}_X)} \widehat{\mathcal{B}ar}_{\bar{w}} \\
\downarrow \scriptstyle 1 \otimes \rho \otimes 1 \wr & & \downarrow \scriptstyle ev \overline{\otimes} 1 \otimes 1 \\
\mathcal{E}^\vee \otimes \mathcal{E} & & \Delta^* \widehat{\mathcal{B}ar}_{\bar{w}} \\
& & \downarrow \scriptstyle HKR \\
& & \Omega_{dw}
\end{array}$$

From lemmas 5.2 and 5.4 we may complete this diagram to the following picture

$$\begin{array}{ccccc}
\mathcal{E}^\vee \otimes (\widehat{B}_{\bar{w}} \otimes_{S(\mathcal{O}_X)} \mathcal{E}) & \xrightarrow{1 \boxtimes \sigma \otimes 1} & \mathcal{E}^\vee \otimes \mathcal{E} \otimes_{S(\mathcal{O}_X)} \widehat{\mathcal{B}ar}_{\bar{w}} & & \\
\downarrow 1 \boxtimes \rho \otimes 1 & \searrow 1 \boxtimes \pi \otimes 1 & \downarrow ev \otimes 1 \otimes 1 & & \\
\mathcal{E}^\vee \otimes \mathcal{E} & \xleftarrow{\sim} \mathcal{E}^\vee \otimes \mathcal{A}t(\mathcal{E}) & \Delta^* \widehat{\mathcal{B}ar}_w & & \\
\downarrow & \downarrow & \downarrow HKR & & \\
\mathcal{E}^\vee \otimes (\Omega_{dw} \otimes \mathcal{E}) & \xrightarrow{\quad} & \Omega_{dw} & & \\
\downarrow & \downarrow & \downarrow & & \\
\mathcal{H}om(\mathcal{E}, \mathcal{E}) & \xrightarrow{\quad} \mathcal{H}om(\mathcal{E}, \Omega_{dw} \otimes \mathcal{E}) \xrightarrow{str} \Omega_{dw} & & &
\end{array}$$

It is easy to check that everything commutes, at least perhaps with the added observation that the bottom arrow, which we have by abuse simply called str , first commutes the tensors under the isomorphism

$$\mathcal{H}om(\mathcal{E}, \Omega_{dw} \otimes \mathcal{E}) \cong \mathcal{H}om(\mathcal{E}, \mathcal{E}) \otimes \Omega_{dw}$$

before applying the super-trace. \square

Taking $w = 0$, our results give us directly information about \mathbb{Z}_2 complexes of vector bundles on X .¹ Moreover, one checks (essentially by taking a standard tensor product on complexes, rather than the folded tensor we use for matrix factorizations) that all of the above constructions and theorem go through. This gives the following result, which follows formally from [1] and [11]. However, there seems to be a problem in Ramadoss's proof in [11]: in the proof of Proposition 2 he uses without explaining the coincidence of the two versions of the Chern character of \mathcal{O}_Δ , one defined in [1] and the one coming from DG theory.

Theorem 5.7. *The DG Chern Character map for perfect complexes on smooth X in the sense of [14] coincides with the classical Chern Character.*

Proof. We apply our remark 5 to lemma 5.6 and 5.1 to find that the DG Chern Character of a bounded complex of locally free sheaves \mathcal{E} is given by

$$\mathbb{R}\Gamma(str(\mathcal{E}xp(at(E)))) \in \mathbb{R}\Gamma(\bigoplus \Omega^i) = HH(X).$$

This according to [3] is exactly the classical Chern Character. \square

6. A FORMULA FOR THE BOUNDARY-BULK MAP

In this section we wish to develop a global analog of the Chern character formula for global matrix factorizations computed for a formal disk in [6]. There should be some question about what such an analog could be since globality generally prohibits formulas, at least formulas involving coordinates. Another option would be to relate to Chern character to certain classes which exist globally, e.g. Chern classes or the Atiyah class. At some level we have already done this and at another we have already discussed

¹There is a mild issue here with respect to 3.10 which requires that w not be a zero divisor, however the conclusion of this theorem is well-known to still hold when $w = 0$, so there is no problem.

the obstruction to doing so. We should probably also point out here that we do not know what Chern classes are for matrix factorizations.

We have taken the task of finding a global Chern character formula and more generally the boundary bulk map to mean the following: understand the image of the boundary bulk map in some computable model for $\mathbb{R}\Gamma(\Omega_{dw})$. This will be a Čech model and we will give our formula in terms of local connections on a Čech cover.

Lemma 6.1. *Let \mathcal{E} be a matrix factorization with curved differential e . Suppose ∇ is a connection on \mathcal{E} , i.e. ∇ consists of standard connections on underlying graded components $\nabla_i : \mathcal{E}_0 \rightarrow \Omega \otimes \mathcal{E}_i$ for $i = 0, 1$. Then the morphism*

$$\begin{array}{ccc} & \mathcal{A}t(\mathcal{E}) & \\ \swarrow \wr & & \searrow \\ \mathcal{E} & & \Omega_{dw} \otimes \mathcal{E} \end{array}$$

which represents the map $\mathcal{E}xp(at(\mathcal{E}))$ in the coderived category from definition 5.5 is given by the map of w -curved complexes

$$\mathcal{E}xp(at(\mathcal{E})) = \sum_{i=0}^n \frac{\wedge[\nabla, e]^i}{i!}.$$

Proof. Given a connection ∇ on \mathcal{E} we obtain splittings $\mathcal{J}^1(\mathcal{E}_i) = \Omega^1 \otimes \mathcal{E}_i \oplus \mathcal{E}_i$ under these splittings the induced map by e on $\mathcal{J}^1(\mathcal{E})$ becomes

$$\begin{pmatrix} 1 \otimes e & [\nabla, e] \\ 0 & e \end{pmatrix},$$

The map

$$m : \Omega^{\otimes q+1} \otimes \mathcal{E}_i \oplus \Omega^q \otimes \mathcal{E}_i \rightarrow \Omega^q \otimes \mathcal{E}_i \oplus \Omega^{q-1} \mathcal{E}_i$$

is simply given by the projection

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and the map B_{dw} splits as

$$\begin{pmatrix} B'_{dw} & 0 \\ 0 & B''_{dw} \end{pmatrix}$$

where $B'_{dw} : \Omega^{\otimes q+1} \otimes \mathcal{E}_i \mapsto \Omega^{\otimes(q+2)} \otimes \mathcal{E}_i$ is given by

$$B'_{dw}(a_0 da_1 \otimes \cdots \otimes da_{q+1} \otimes e) = \sum_{j=0}^q (-1)^j a_0 da_1 \otimes \cdots \otimes da_j \otimes dw \otimes da_{j+1} \otimes \cdots \otimes da_{q+1} \otimes e$$

and $B''_{dw} : \Omega^{\otimes q} \otimes \mathcal{E}_i \mapsto \Omega^{\otimes(q+1)} \otimes \mathcal{E}_i$ is given by

$$B''_{dw}(a_0 da_1 \otimes \cdots \otimes a_q \otimes e) = \sum_{j=0}^q (-1)^j a_0 da_1 \otimes \cdots \otimes da_j \otimes dw \otimes da_{j+1} \otimes \cdots \otimes da_q \otimes e.$$

Using these splittings we may view $\sum_{i=0}^n [\nabla, e]^i$ as a degree 0 map $\mathcal{E} \rightarrow \mathcal{A}t(\mathcal{E})$ then it will suffice to show that that map is a closed morphism of w -curved modules since it obviously splits the weak-equivalence $\mathcal{A}t(\mathcal{E}) \rightarrow \mathcal{E}$ induced by projecting and we also have

$$\left(\sum_i \frac{\wedge}{i!} \right) \circ \left(\sum_i [\nabla, e]^i \right) = \mathcal{E}xp(at(\mathcal{E})).$$

We first need to make the simple calculation:

$$e[\nabla, e] + [\nabla, e]e = \nabla w - w\nabla = dw.$$

Now

$$\begin{aligned} e\gamma[\nabla, e]^q - [\nabla, e]^qe &= (-1)^qe[\nabla, e]^q - [\nabla, e]^qe \\ &= (-1)^q \sum_{i=0}^{q-1} (-1)^i [\nabla, e]^i (e[\nabla, e] + [\nabla, e]e) [\nabla, e]^{q-i-1} \\ &= - \sum_{i=0}^{q-1} (-1)^{q-i-1} [\nabla, e]^i \otimes dw \otimes [\nabla, e]^{q-i-1} \\ &= -B''_{dw} [\nabla, e]^{q-1} \end{aligned}$$

The equality going from line 3 to 4 above is a bit tricky: The map $dw \otimes -$ sends

$$a_0 da_1 \otimes da_{q-i-1} \otimes e \mapsto a_0 da_1 \otimes da_{q-i-1} \otimes dw \otimes e$$

and then for the composition we have

$$\begin{array}{ccc} \mathcal{E}_j & \xrightarrow{[d, \nabla]^{q-i-1}} & \Omega^{\otimes q-i-1} \otimes \mathcal{E}_{j+q-i-1} \\ & & \downarrow dw \otimes - \\ & & \Omega^{\otimes q-i-1} \otimes \Omega^1 \otimes \mathcal{E}_{j+q-i-1} \xrightarrow{[d, \nabla]^i} \Omega^{\otimes q-i-1} \otimes \Omega^1 \otimes \Omega^i \otimes \mathcal{E}_{j+q-1} \end{array}$$

so computing $[\nabla, e]^i \otimes dw \otimes [e, \nabla]^{q-i-1}$ is the same as computing $[\nabla, e]^{q-1}$ and then inserting dw in the $q-i-1$ st slot. Now we may compute

$$\begin{aligned} & \left[\begin{pmatrix} B'_{dw} & 0 \\ 0 & B''_{dw} \end{pmatrix} - \begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} d\mathcal{E}\gamma & [\nabla, e]\gamma \\ 0 & e\gamma \end{pmatrix} \right] \begin{pmatrix} 0 \\ \sum_{i=0}^n [\nabla, e]^i \end{pmatrix} \\ &= \sum_{i=0}^n \begin{pmatrix} 0 \\ B''_{dw} [\nabla, e]^i \end{pmatrix} - \sum_{i=1}^n \begin{pmatrix} (-1)^i [\nabla, e]^i \\ 0 \end{pmatrix} \\ & \quad + \sum_{i=0}^n \begin{pmatrix} 0 \\ (-1)^i e [\nabla, e]^i \end{pmatrix} + \sum_{i=0}^n \begin{pmatrix} (-1)^i [\nabla, e]^{i+1} \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ \sum_{i=0}^n [\nabla, e]^i e \end{pmatrix} \end{aligned}$$

which finishes the proof. Note that the second sum on the second line starts at $i = 1$ because the component of the differential on $\mathcal{A}t(E)$ coming from m is 0 on $\mathcal{J}^1(\mathcal{E})$. \square

Remark 6.2. In the case when X is a formal disk, we recover the formula for the Chern Character from [6] by identifying the cohomology $\mathbb{R}\Gamma(\Omega_X)$ with the Tyurina algebra .

Naturality of the Chern character (or more generally the boundary bulk map) implies that it commutes with restriction to open subschemes. The above lemma tells us what happens to the Chern Character upon restriction to open affine subschemes. Of course, upon restricting we loose information. The following lemmas describe how we can go the other direction. That is, they give us a method to take this local data (along with an appropriate collection of homotopies) to a global a global morphism to Cech Cohomology.

Lemma 6.3. *Let $(\mathcal{F}, d_{\mathcal{F}})$ and $(\mathcal{G}, d_{\mathcal{G}})$ be complexes of sheaves (or w -curved $S(\mathcal{O}_X)$ modules) on X and U_1, \dots, U_n be a Cech cover of X . Denote $\mathcal{G}_{i_0 \dots i_p} = (j_{i_0 \dots i_p})_* \mathcal{G}|_{U_{i_0} \cap \dots \cap U_{i_p}}$, where $j_{i_0 \dots i_p}$ is the inclusion of $U_{i_0} \cap \dots \cap U_{i_p}$ into X . Suppose we are given the following*

data: for each $0 \leq p \leq n$ and each tuple $i_0 i_1 \dots i_p$ with $1 \leq i_0 < i_1 < \dots < i_p \leq n$ we have a map

$$f_{i_0 \dots i_p} : \mathcal{F} \rightarrow \mathcal{G}_{i_0 \dots i_p}[p]$$

such that

$$d_{\mathcal{G}} f_{i_0 \dots i_p} - (-1)^p f_{i_0 \dots i_p} d_{\mathcal{F}} = \sum_{j=0}^p (-1)^k f_{i_0 \dots \hat{i}_k \dots i_p} |_{U_{i_0 \dots i_p}},$$

then the map $f : \mathcal{F} \rightarrow \text{Cech}(\mathcal{G})$ defined on \mathcal{F}^q by

$$f = \sum_p \sum_{i_0 \dots i_p} (-1)^{\frac{p(p-1)}{2}} f_{i_0 \dots i_p}$$

is a closed degree 0 map of complexes.

Proof. First observe that $f_{i_0 \dots i_p}$ takes \mathcal{F}^q to $\mathcal{G}_{i_0 \dots i_p}^{q-p}$ and $\mathcal{G}_{i_0 \dots i_p}^{q-p}$ lives in degree q of $\text{Cech}(\mathcal{G})$, therefore the map f is indeed degree 0.

If we consider the composition cf where c is the Cech differential, we get

$$\begin{aligned} cf &= \sum_{p=0}^n \sum_{i_0 \dots i_p} (-1)^{\frac{p(p-1)}{2}} c f_{i_0 \dots i_p} \\ &= \sum_{p=0}^n \sum_{\substack{\mathbf{i}=i_0 \dots i_p \\ \mathbf{j}=j_0 \dots j_{p+1}}} (-1)^{\frac{p(p-1)}{2}} \sigma(\mathbf{i}, \mathbf{j}) f_{i_0 \dots i_p} |_{U_{j_0 \dots j_{p+1}}} \end{aligned}$$

where

$$\sigma(\mathbf{i}, \mathbf{j}) = \begin{cases} 1 & \text{if } i_0 \dots i_p = j_0 \dots \hat{j}_k \dots j_{p+1}, \text{ } k \text{ even} \\ -1 & \text{if } i_0 \dots i_p = j_0 \dots \hat{j}_k \dots j_{p+1}, \text{ } k \text{ odd} \\ 0 & \text{else} \end{cases}$$

On the other hand, by our assumption on $f_{i_0 \dots i_p}$, we have

$$\begin{aligned} \gamma d_{\mathcal{G}} f - f d_{\mathcal{F}} &= \sum_{p=0}^n \sum_{i_0 \dots i_p} (-1)^{\frac{p(p-1)}{2}+p} (d_{\mathcal{G}} f_{i_0 \dots i_p} - (-1)^p f_{i_0 \dots i_p} d_{\mathcal{F}}) \\ &= \sum_{p=0}^n \sum_{i_0 \dots i_p} (-1)^{\frac{p(p-1)}{2}+p} \sum_{k=0}^p (-1)^k f_{i_0 \dots \hat{i}_k \dots i_p} |_{U_{i_0 \dots i_p}} \\ &= \sum_{p=0}^n \sum_{\substack{\mathbf{i}=i_0 \dots i_p \\ \mathbf{j}=j_0 \dots j_{p+1}}} (-1)^{\frac{(p+1)(p)}{2}+p+1} \sigma(\mathbf{i}, \mathbf{j}) f_{i_0 \dots i_p} |_{U_{j_0 \dots j_{p+1}}} \\ &= - \sum_{p=0}^n \sum_{\substack{\mathbf{i}=i_0 \dots i_p \\ \mathbf{j}=j_0 \dots j_{p+1}}} (-1)^{\frac{p(p-1)}{2}} \sigma(\mathbf{i}, \mathbf{j}) f_{i_0 \dots i_p} |_{U_{j_0 \dots j_{p+1}}} \\ &= -cf \end{aligned}$$

where γ is the grading operator on the Cech complex: $\gamma|_{\mathcal{G}_{i_0 \dots i_p}} = (-1)^p$. \square

Lemma 6.4. Let \mathcal{E} be a matrix factorization with curved differential e . Let ∇_j be a choice of connection on U_j , where $\{U_j\}_{j=1}^N$ is a Cech cover of X . The collection of maps

$$f_{i_0 \dots i_p} : \mathcal{E} \rightarrow \mathcal{A}t(\mathcal{E})_{i_0 \dots i_p}[p],$$

$$f_{i_0 \dots i_p} = \sigma_{i_0} \sum_{k_0, k_1, \dots, k_p} \tau_p(k_0, \dots, k_p) [e, \nabla_{i_0}]^{k_0} (\nabla_{i_0} - \nabla_{i_1}) [e, \nabla_{i_1}]^{k_1} (\nabla_{i_1} - \nabla_{i_2}) \dots [e, \nabla_{i_p}]^{k_p}$$

$\tau_p(k_0, \dots, k_p) = (-1)^{\sum_{j=0}^p j(k_j+1)}$ and $\sigma_{i_0} : \Omega^{\otimes q} \otimes \mathcal{E} \rightarrow \Omega^{\otimes q} \otimes \mathcal{J}^1(\mathcal{E})$ is the splitting induced by ∇_{i_0} satisfies the hypothesis of lemma 6.3.

Proof. We know from the proof of lemma 6.1 that

$$(-1)^k e [\nabla_{i_0}, e] - [\nabla_{i_0}, e]^k e = -B_{dw} [\nabla_{i_0}, e]^{k-1}$$

so that

$$\gamma e f_{i_0} - f_{i_0} e = -B_{dw} f_{i_0}.$$

We claim that

$$e f_{i_0 \dots i_p} - (-1)^p f_{i_0 \dots i_p} e = -B_{dw} + \sum_{j=0}^p (-1)^j f_{i_0 \dots \hat{i}_j \dots i_p}.$$

We will prove this by induction, but before we do, let us see how this proves the lemma.

Recall from the proof of lemma 6.1 that after splitting $\mathcal{A}t(\mathcal{E})$ with respect to ∇_{i_0} the differential is given as the sum of three components

$$\gamma e = (-1)^q \begin{pmatrix} 1 \otimes e & [\nabla_{i_0}, e] \\ 0 & e \end{pmatrix} : \Omega^{\otimes q+1} \otimes \mathcal{E}_i \oplus \Omega^q \otimes \mathcal{E}_i \rightarrow \Omega^{q+1} \otimes \mathcal{E}_{i+1} \oplus \Omega^q \otimes \mathcal{E}_{i+1}$$

$$-\gamma m = (-1)^q \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} : \Omega^{q+1} \otimes \mathcal{E}_i \oplus \Omega^q \otimes \mathcal{E}_i \rightarrow \Omega^q \otimes \mathcal{E}_i \oplus \Omega^{q-1} \otimes \mathcal{E}_i$$

and

$$B_{dw} = \begin{pmatrix} B'_{dw} & 0 \\ 0 & B''_{dw} \end{pmatrix} : \Omega^{q+1} \otimes \mathcal{E}_i \oplus \Omega^q \otimes \mathcal{E}_i \rightarrow \Omega^{q+2} \otimes \mathcal{E}_i \oplus \Omega^{q+1} \otimes \mathcal{E}_i.$$

We have the relation

$$\begin{aligned} & [\nabla_{i_0}, e] f_{i_0 \dots i_p} - f_{i_0 \dots i_p} \\ &= - \sum_{k_1 \dots k_p} (-1)^{\sum_{j=1}^p j(k_j+1)} (\nabla_{i_0} - \nabla_{i_1}) [\nabla_{i_1}, e]^{k_1} \dots [\nabla_{i_p}, e]^{k_p} \\ &= (\nabla_{i_1} - \nabla_{i_0}) \sum_{k_1 \dots k_p} (-1)^{k_1 + \dots + k_p + p-1} (-1)^{\sum_{j=0}^p j(k_j+1)} (\nabla_{i_0} - \nabla_{i_1}) [\nabla_{i_1}, e]^{k_1} \dots [\nabla_{i_p}, e]^{k_p} \end{aligned}$$

So when we take into account the grading operator γ we get

$$(\gamma [\nabla_{i_0}, e] - \gamma) f_{i_0 \dots i_p} = (\nabla_{i_1} - \nabla_{i_0}) f_{i_1 \dots i_p}.$$

Now the observation is that $(\nabla_{i_1} - \nabla_{i_0}) f_{i_1 \dots i_p}$ is exactly the difference between splitting with respect to ∇_{i_1} and splitting with respect to ∇_{i_0} i.e.

$$\sigma_{i_0} f_{i_1 \dots i_p} + (\nabla_{i_1} - \nabla_{i_0}) f_{i_1 \dots i_p} = \sigma_{i_1} f_{i_1 \dots i_p}$$

This combined with the claim gives us the lemma.

To prove the claim, notice first that we can write

$$f_{i_0 \dots i_p} = \sum_{k=0}^n (-1)^{kp+p} f_{i_0 \dots i_{p-1}} (\nabla_{i_{p-1}} - \nabla_{i_p}) [\nabla_{i_p}, d]^k.$$

Then we make the computation:

$$\begin{aligned}
\gamma e f_{i_0 \dots i_p} - (-1)^p f_{i_0 \dots i_p} e &= \sum_k (-1)^{kp+p+k+1} (\gamma e f_{i_0 \dots i_{p-1}} - (-1)^{p-1} f_{i_0 \dots i_{p-1}}) (\nabla_{i_{p-1}} - \nabla_{i_p}) [\nabla_{i_p}, e]^k \\
&\quad + \sum_k (-1)^{kp+k} f_{i_0 \dots i_{p-1}} ([\nabla_{i_p}, e] - [\nabla_{i_{p-1}}, e]) [\nabla_{i_p}, e]^k \\
&\quad + \sum_k (-1)^{kp} f_{i_0 \dots i_{p-1}} (\nabla_{i_{p-1}} - \nabla_{i_p}) ((-1)^k e [\nabla_{i_p}, e]^k - [\nabla_{i_p}, e]^k e) \\
(7) \quad &= \sum_k (-1)^{kp+k+p+1} (-B_{dw} f_{i_0 \dots i_{p-1}}) (\nabla_{i_{p-1}} - \nabla_{i_p}) [\nabla_{i_p}, e]^k \\
(8) \quad &\quad + \sum_k (-1)^{(k+1)(p-1)} \left(\sum_{j=0}^{p-1} (-1)^j f_{i_0 \dots \widehat{i_j} \dots i_{p-1}} \right) (\nabla_{i_{p-1}} - \nabla_{i_p}) [\nabla_{i_p}, e]^k \\
(9) \quad &\quad + \sum_k (-1)^{kp+k} f_{i_0 \dots i_{p-1}} ([\nabla_{i_p}, e] - [\nabla_{i_{p-1}}, e]) [\nabla_{i_p}, e]^k \\
(10) \quad &\quad + \sum_k (-1)^{kp} f_{i_0 \dots i_{p-1}} (\nabla_{i_{p-1}} - \nabla_{i_p}) (-B_{dw} [\nabla_{i_p}, e]^{k-1})
\end{aligned}$$

Note that the sign $(-1)^{kp+k+p+1}$ on the first line appears because applying $(\nabla_{i_{p-1}} - \nabla_{i_p}) [\nabla_{i_p}, e]^k$ before $f_{i_0 \dots i_{p-1}}$ introduces an extra $k+1$ tensor factors of Ω^1 . So what we will need to show is that lines (7), (8), (9) and (10) sum to give

$$-B_{dw} f_{i_0 \dots i_p} + \sum_{j=0}^p (-1)^j f_{i_0 \dots \widehat{i_j} \dots i_p}.$$

Now we note that

$$\begin{aligned}
&(-1)^{k+1} (-B_{dw} f_{i_0 \dots i_{p-1}}) (\nabla_{i_{p-1}} - \nabla_{i_p}) [\nabla_{i_p}, e]^k \\
&\quad + f_{i_0 \dots i_{p-1}} (\nabla_{i_{p-1}} - \nabla_{i_p}) (-B_{dw} [\nabla_{i_p}, e]^{k-1}) \\
&\quad = B_{dw} f_{i_0 \dots i_{p-1}} (\nabla_{i_{p-1}} - \nabla_{i_p}) [\nabla_{i_p}, e]^k
\end{aligned}$$

so the sums

$$\sum_k (-1)^{kp} f_{i_0 \dots i_{p-1}} (\nabla_{i_{p-1}} - \nabla_{i_p}) (-B_{dw} [\nabla_{i_p}, e]^{k-1})$$

from (10) and

$$\sum_{k=0}^{\infty} (-1)^{kp+k+p+1} (-B_{dw} f_{i_0 \dots i_{p-1}}) (\nabla_{i_{p-1}} - \nabla_{i_p}) [\nabla_{i_p}, e]^k$$

from (7) add to give $-B_{dw} f_{i_0 \dots i_p}$ as needed.

We have the following relation

$$f_{i_0 \dots i_{p-1}} [\nabla_{i_{p-1}}, e] = (-1)^{p-1} f_{i_0 \dots i_{p-1}} - f_{i_0 \dots i_{p-2}} (\nabla_{i_{p-2}} - \nabla_{i_{p-1}})$$

so then for the sum from (9) we have

$$\begin{aligned}
& \sum_k (-1)^{kp+k} f_{i_0 \dots i_{p-1}} ([\nabla_{i_p}, e] - [\nabla_{i_{p-1}}, e]) [\nabla_{i_p}, e]^k \\
&= \sum_k (-1)^{kp+k} f_{i_0 \dots i_{p-1}} [\nabla_{i_p}, e]^{k+1} \\
&\quad - \sum_k (-1)^{kp+k} f_{i_0 \dots i_{p-1}} [\nabla_{i_{p-1}}, e] [\nabla_{i_p}, e]^k \\
&= \sum_k (-1)^{kp+k} f_{i_0 \dots i_{p-1}} [\nabla_{i_p}, e]^{k+1} \\
&\quad + \sum_k (-1)^{kp+k+p} f_{i_0 \dots i_{p-1}} [\nabla_{i_p}, e]^k \\
&\quad + \sum_k (-1)^{kp+k+1} f_{i_0 \dots i_{p-2}} (\nabla_{i_{p-2}} - \nabla_{i_{p-1}}) [\nabla_{i_p}, e]^k \\
(11) \quad &= (-1)^p f_{i_0 \dots i_{p-1}} \\
(12) \quad &+ \sum_k (-1)^{kp+k+1} f_{i_0 \dots i_{p-2}} (\nabla_{i_{p-2}} - \nabla_{i_{p-1}}) [\nabla_{i_p}, e]^k
\end{aligned}$$

Now we turn our attention to the sum

$$\sum_{k=0}^n (-1)^{(k+1)(p-1)} \left(\sum_{j=0}^{p-1} (-1)^j f_{i_0 \dots \widehat{i_j} \dots i_{p-1}} \right) (\nabla_{i_{p-1}} - \nabla_{i_p}) [\nabla_{i_p}, e]^k$$

from (8) In the case when $j \neq p-1$ we have

$$(13) \quad \sum_{k=0}^{\infty} (-1)^{(k+1)(p-1)} f_{i_0 \dots \widehat{i_j} \dots i_{p-1}} (\nabla_{i_{p-1}} - \nabla_{i_p}) [\nabla_{i_p}, e]^k = f_{i_0 \dots \widehat{i_j} \dots i_p}$$

When $j = p-1$ we may add

$$\sum_{k=0}^n (-1)^{(k+1)(p-1)+p-1} f_{i_0 \dots i_{p-2}} (\nabla_{i_{p-1}} - \nabla_{i_p}) [\nabla_{i_p}, e]^k$$

to

$$\sum_{k=0}^n (-1)^{kp+k+1} f_{i_0 \dots i_{p-2}} (\nabla_{i_{p-2}} - \nabla_{i_{p-1}}) [\nabla_{i_p}, e]^k$$

from (12) to get

$$(14) \quad (-1)^{p-1} \sum_{k=0}^{\infty} (-1)^{k(p-1)} f_{i_0 \dots i_{p-2}} (\nabla_{i_{p-2}} - \nabla_{i_p}) [\nabla_{i_p}, e]^k = (-1)^{p-1} f_{i_0 \dots \widehat{i_{p-1}} i_p}.$$

Adding the sums from (11), (13) and (14) gives

$$\sum_{j=0}^p (-1)^j f_{i_0 \dots \widehat{i_j} \dots i_p}$$

which finishes the claim and thus the lemma. □

Theorem 6.5. *Let $f = \{f_{i_0 \dots i_p}\} \in \text{Cech}(\mathcal{H}om(\mathcal{E}, \mathcal{E}))$. The boundary bulk map $\tau_{\mathcal{E}}$ is computed on f as*

$$f_{i'_0 \dots i'_q} \mapsto \text{str} \left(\sum_p \sum_{i_0 \dots i_p} \sum_{k_1 \dots k_p} (-1)^{p+\sum_{j=0}^p j k_j} \frac{[\nabla_{i_0}, e]^{k_0} (\nabla_{i_0} - \nabla_{i_1}) \dots (\nabla_{i_{p-1}} - \nabla_{i_p}) [\nabla_{i_p}, e]^{k_p}}{(k_0 + \dots + k_p + p)!} \circ f_{i'_0 \dots i'_q} \right)$$

Proof. This is mostly an amalgamation of lemmas 6.3, 6.4 and 5.6. The division by $(k_0 + \dots k_p + p)!$ comes from applying the map

$$\sum \frac{\wedge}{i!} : Cech(\mathcal{A}t(\mathcal{E})) \rightarrow Cech(\Omega_{dw} \otimes \mathcal{E})$$

(see lemma 5.4). The sign comes from the fact that

$$\frac{p(p-1)}{2} + \sum_{j=0}^p j(k_j + 1) = \frac{p(p-1)}{2} + \frac{p(p+1)}{2} + \sum_{j=0}^p jk_j = p^2 + \sum_{j=0}^p jk_j$$

and p is congruent to p^2 modulo 2. We need only check that this map we have constructed actually computes the boundary bulk-map.

We have the following diagram

$$\begin{array}{ccc} \mathcal{A}t(\mathcal{E}) & \longrightarrow & Cech(\mathcal{A}t(\mathcal{E})) \\ \pi \downarrow & \nearrow & \downarrow Cech(\pi) \\ \mathcal{E} & \longrightarrow & Cech(\mathcal{E}) \end{array}$$

where the diagonal map is given by

$$(15) \quad \sum_p \sum_{i_0 \dots i_p} \sum_{k_1 \dots k_p} (-1)^{p^2 + \sum_{j=0}^p jk_j} [\nabla_{i_0}, e]^{k_0} (\nabla_{i_0} - \nabla_{i_1}) \dots (\nabla_{i_{p-1}} - \nabla_{i_p}) [\nabla_{i_p}, e]^{k_p}.$$

Now the outside square commutes as well as the bottom right triangle. And then, since $Cech(\pi)$ is a weak equivalence, the upper left triangle commutes in the coderived category. It follows then that composition of the diagonal map, (15), with the map $\sum_i \frac{\wedge}{i!} : Cech(\mathcal{A}t(\mathcal{E})) \rightarrow Cech(\Omega_{dw})$ computes $\iota \mathcal{E}xp(\mathcal{A}t(\mathcal{E}))$, where $\iota : \Omega_{dw} \rightarrow Cech(\Omega_{dw})$ is the inclusion. \square

Corollary 6.6. *The Chern Character \mathcal{E} is given by the cocycle*

$$ch(\mathcal{E}) = str \left(\sum_p \sum_{i_0 \dots i_p} \sum_{k_1 \dots k_p} (-1)^{p + \sum_{j=0}^p jk_j} \frac{[\nabla_{i_0}, e]^{k_0} (\nabla_{i_0} - \nabla_{i_1}) \dots (\nabla_{i_{p-1}} - \nabla_{i_p}) [\nabla_{i_p}, e]^{k_p}}{(k_0 + \dots + k_p + p)!} \right)$$

in a Cech model for $\mathbb{R}\Gamma(\Omega_{dw})$.

Remark 6.7. In light of remark 5 and theorem 5.7, corollary 6.6 translates directly to give a formula for the Chern character of complexes of vector bundles.

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